“Parametric Lambda Calculus”

Simona Ronchi Della Rocca

Luca Paolini

ronchi@di.unito.it

Università di Torino

Dipartimento di Informatica

Outline

- Historical Introduction

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- The syntax of the parametric λ calculus

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- Crossing Call-by-name and Call-by-value

A bit of history

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The translation of Algol 60 into λ-calculus (Landin)
The language CUCH (Böhm and Gross)
The Separability Theorem (Böhm)

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The Separability Theorem (Böhm)

~ 1970 The denotational semantics of λ-calculus (Scott)
The λ-β_v-calculus (Plotkin)

A bit of history

$80 - 90$ The incompleteness of λ-calculus (Ronchi, Honsell)
The full-abstraction problem (Plotkin, Berry)
The lazy λ-calculus (Abramsky)
Other λ-calculi....

Some λ-calculi

- The classical λ-calculus:
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- answer:
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- It allows to generalize definitions, properties and proofs
- It is a natural starting point for the study of new calculi
- It helps the investigations of the relations between different calculi

The Syntax

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$$M ::= x \mid MM \mid \lambda x.M$$

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- Let $\Delta \subseteq \Lambda$. The Δ-reduction ($\rightarrow_\Delta$) is the contextual closure of the following rule:

$$(\lambda x.M)N \rightarrow_\Delta M[N/x] \text{ if and only if } N \in \Delta$$

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- The above requirements assure that the status of being an input value is preserved during the evalution process.

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- Λ_I, defined by the following grammar:

$$M ::= x \mid MM \mid \lambda x.M (x \in FV(M))$$

  is a set of input values

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- Γ-normal forms are not input values

Confluence

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Confluence

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- The closure conditions on the set Δ of input values assure that the corresponding λ-Δ-calculus enjoys the confluence property.
- The closure conditions are sufficient but not necessary for garantee the confluence property (instead of the reduction closure it would be sufficent to have: $M \in \Delta$ and $M \to^*_\Delta N$ imply there is $P \in \Delta$ such that $N \to^*_\Delta P$.)

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A counterexample

Let us take Λ-normal forms as input values.

Then the confluence fails, in fact:

$(\lambda x.(\lambda y.z)(x(\lambda x.xx)))(\lambda x.xx)$ reduces to both z and $(\lambda y.z)((\lambda x.xx)(\lambda x.xx))$, which do not have a common reduct.

Standardization

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The reduction sequence reducing the same term from left to right in the $\lambda\Gamma$-calculus:

$$(\lambda x.x(KI))(II) \to_{\Gamma} (\lambda x.x(\lambda y.I))(II) \to_{\Gamma} (\lambda x.x(\lambda y.I))I \to_{\Gamma} I(\lambda y.I)$$

is not a standard reduction sequence!

Sequentialization

- The Δ-sequentialization $(M)^\circ$ of a term M is a function from Λ to Λ defined as follows:
  - $(xM_1...M_m)^\circ = x(M_1)^\circ...(M_m)^\circ$;
  - $((\lambda x.P)QM_1...M_m)^\circ = (\lambda x.P)^\circ(Q)^\circ(M_1)^\circ...(M_m)^\circ$ if $Q \in \Delta$;
  - $((\lambda x.P)QM_1...M_m)^\circ = (Q)^\circ(\lambda x.P)^\circ(M_1)^\circ...(M_m)^\circ$ if $Q \notin \Delta$;
  - $(\lambda x.P)^\circ = \lambda x.(P)^\circ$.

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  - $(\lambda x.P)^\circ = \lambda x.(P)^\circ$.
- The degree of a redex R in M is the numbers of λ's which both are active in M and occur on the left of $(R)^\circ$ in $(M)^\circ$.

  (A symbol λ in a term M is active if and only if it is the first symbol of a Δ-redex of M).

Sequentialization

- A sequence $M \equiv P_0 \rightarrow_\Delta P_1 \rightarrow_\Delta ... \rightarrow_\Delta P_n \rightarrow_\Delta N$ is standard if and only if the degree of the redex contracted in P_i is less than or equal to the degree of the redex contracted in P_{i+1}, for every $i < n$.

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- A set Δ of input values is standard if and only if $M \notin \Delta$ and $M \rightarrow^*_\Delta N$ by reducing at every step a not principal redex imply $N \notin \Delta$.
  (The principal redex is the redex of degree 0
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  (The principal redex is the redex of degree 0 corresponding to the head redex in the $\lambda\Lambda$-calculus)
- A $\lambda\Delta$-calculus is standard if and only if the set Δ is standard
- To be standard is a necessary and sufficent condition for enjoying the standardization property.

Standardization (examples)

- $(\lambda x.x(KI))(II) \rightarrow_\Gamma (\lambda x.x(KI))I \rightarrow_\Gamma I(KI) \rightarrow_\Gamma I(\lambda y.I)$

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- The set of input values Λ, Γ, Ξ are standard.
- The set of input values Λ_I is not standard.

Evaluation

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- The principal evaluation is normalizing.
- There is a Parametric Principal Reduction Machine, parametric with respect to the set of input values Δ, reducing a term according to the principal evaluation.

The PPR machine

$$\frac{M \rightarrow^p_\Delta N}{\lambda x.M \rightarrow^p_\Delta \lambda x.N}\ p1 \qquad \frac{i = \min\{j \leq m | M_i \notin \Delta\text{-nf}\} \quad M_i \rightarrow^p_\Delta N_i}{xM_1 \ldots M_m \rightarrow^p_\Delta xM_1 \ldots N_i \ldots M_m}\ p2$$

$$\frac{Q \in \Delta}{(\lambda x.P)QM_1 \ldots M_m \rightarrow^p_\Delta P[Q/x]M_1 \ldots M_m}\ p3$$

$$\frac{Q \notin \Delta \quad Q \notin \Delta\text{-nf} \quad Q \rightarrow^p_\Delta Q'}{(\lambda x.P)QM_1 \ldots M_m \rightarrow^p_\Delta (\lambda x.P)Q'M_1 \ldots M_m}\ p4$$

$$\frac{Q \notin \Delta \quad Q \in \Delta\text{-nf} \quad P \notin \Delta\text{-nf} \quad P \rightarrow^p_\Delta P'}{(\lambda x.P)QM_1 \ldots M_m \rightarrow^p_\Delta (\lambda x.P')QM_1 \ldots M_m}\ p5$$

$$\frac{Q \notin \Delta \quad P, Q \in \Delta\text{-nf} \quad i = \min\{j \leq m | M_i \notin \Delta\text{-nf}\} \quad M_i \rightarrow^p_\Delta N_i}{(\lambda x.P)QM_1 \ldots M_m \rightarrow^p_\Delta (\lambda x.P)QM_1 \ldots N_i \ldots M_m}\ p6$$

Output values

- Let Δ be a set of input values.
  A set of output values with respect to Δ is any set $\Theta \subseteq \Lambda$ such that:
  - Θ contains all the Δ-normal forms;
  - If $M =_{\Delta} N$ and $N \in \Theta$ then there is $P \in \Theta$ such that $M \to_{\Delta}^{*p} P$

    ($\to^{*p}$ denotes the reduction sequence choosing at every step the principal redex)

Examples of Output Values

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- Γ together with Γ-normal forms are a set of output values w.r.t Γ.

The Operational Semantics

- Let Θ be a set of output values with respect to the set of input value Δ. $\Downarrow_{\Delta,\Theta}$ is the evaluation relation defined through the following rules:

$$\frac{M \in \Theta}{M \Downarrow_{\Delta,\Theta} M}\ (axiom) \qquad\qquad \frac{M \to^p_\Delta P \qquad P \Downarrow_{\Delta,\Theta} N}{M \Downarrow_{\Delta,\Theta} N}\ (eval)$$

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- Operational pre-order:
$M \preceq_{\Delta,\Theta} N$ if and only if, for all contexts $C[.]$ such that $C[M], C[N] \in \Lambda^0$, $\big(C[M] \Downarrow_{\Delta,\Theta}$ implies $C[N] \Downarrow_{\Delta,\Theta}\big)$.

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- Operational Equivalence:
  $M \approx_{\Delta,\Theta} N$ if and only if $M \preceq_{\Delta,\Theta} N$ and $N \preceq_{\Delta,\Theta} M$.

Normal forms: an apparent paradox

- In the $\lambda\Lambda$-calculus, the set of normal forms:

$$M ::= x \mid \lambda\vec{x}.xM...M$$

 is semantically meaningful. The Böhm separability property holds, so two $\beta\eta$-different normal forms cannot be equated.

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- In a generic $\lambda\Delta$-calculus, the set of normal forms:

$$M ::= x \mid \lambda\vec{x}.xM...M \mid \lambda\vec{x}.(\lambda x.P)QM..M$$

$$(Q \notin \Delta)$$

 has not good semantical properties.

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- A Separability Theorem holds, saying that two different Λ-normal forms can be Γ-separated, so they are different in every Γ model.
- We conjecture that the same holds for all meaningful Δ.

Solvability & Valuability

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- A term M is potentially Δ-valuable iff there is a substitution $\mathbf{s}$ replacing variables by closed values, s.t. $\mathbf{s}(M)$ is Δ-valuable.
- A term M is Δ-solvable if and only if there is a head Δ-context $C[.] \equiv (\lambda\vec{x}.[.])\vec{N}$ such that:

$\vec{x}$ sequentialize variables in $\mathrm{FV}(M)$

$\vec{N} \subseteq \Delta$ and $C[M] =_{\Delta} I$

Γ-Solvability, Γ and Γ-normal forms

- We already showed that the term:

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where $D \equiv \lambda x.xx$
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- $\lambda x.(\lambda y.D)(xI)D \in \Gamma$!

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- DD is Γ-unsolvable
- $(\lambda y.D)(xI)D$ is Γ-unsolvable!
- $\lambda x.(\lambda y.D)(xI)D \in \Gamma$!
- So neither the set of Γ-normal forms nor the set Γ is not a proper subset of the set of Γ-solvable terms!

Call-by-name, call-by-value relations

Some interesting sets of terms in call-by-value calculi correspond to semantically relevant set of the classical (call-by-name) calculus.

- The set of potentially Γ-valuable terms coincides with the set of lazy Λ-strongly normalizing terms.

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- The set of potentially Γ-valuable terms coincides with the set of lazy Λ-strongly normalizing terms.
- The set of Φ-solvable terms coincides with the set of Λ-strongly normalizing

  (The $\lambda\Phi$-calculus is a not lazy call-by-value calculus.)

Lazy Reductions

Let $\Delta \subseteq \Lambda$ be a set of terms.

- The lazy Δ-reduction ($\rightarrow_{\Delta\ell}$) is the applicative closure of the following rule:

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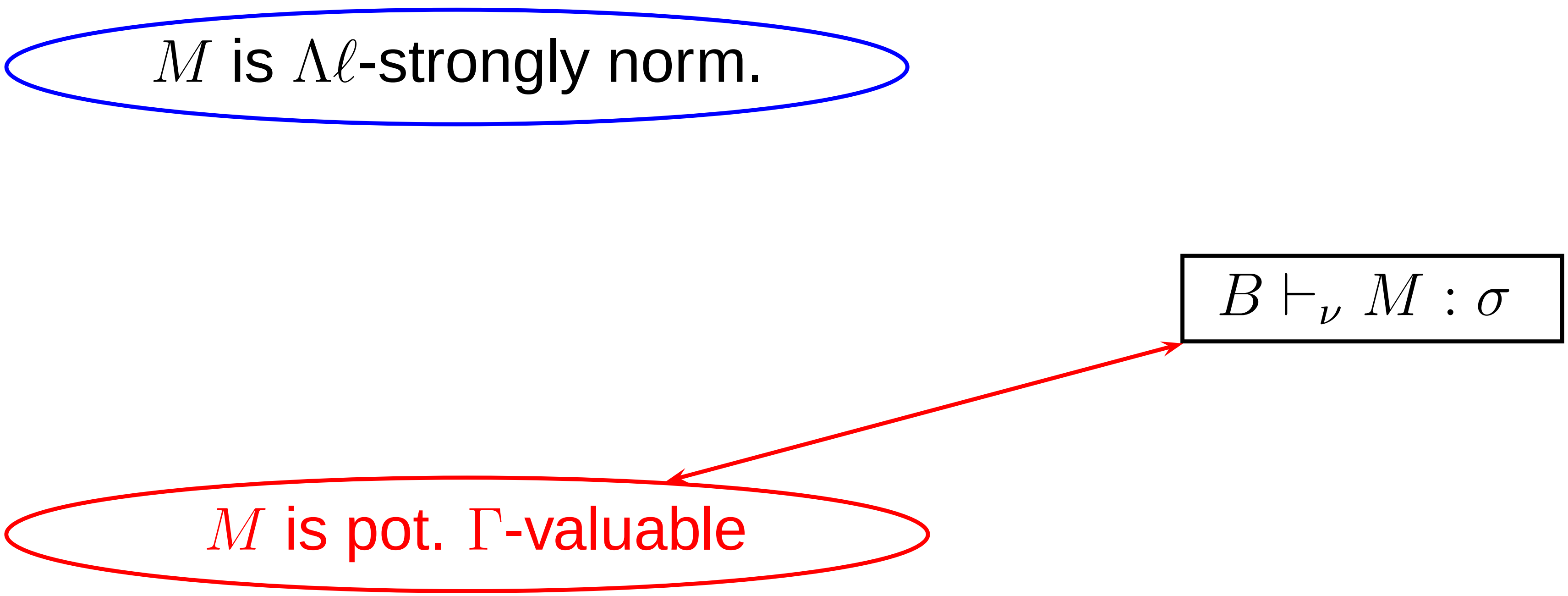
- Let $I \equiv \lambda x.x$; then $\lambda x.II \rightarrow_{\Lambda} \lambda x.I$
  while $\lambda x.II \not\rightarrow_{\Lambda\ell} \lambda x.I$

Γ-pot. valuability and $\Lambda\ell$-strongly norm.



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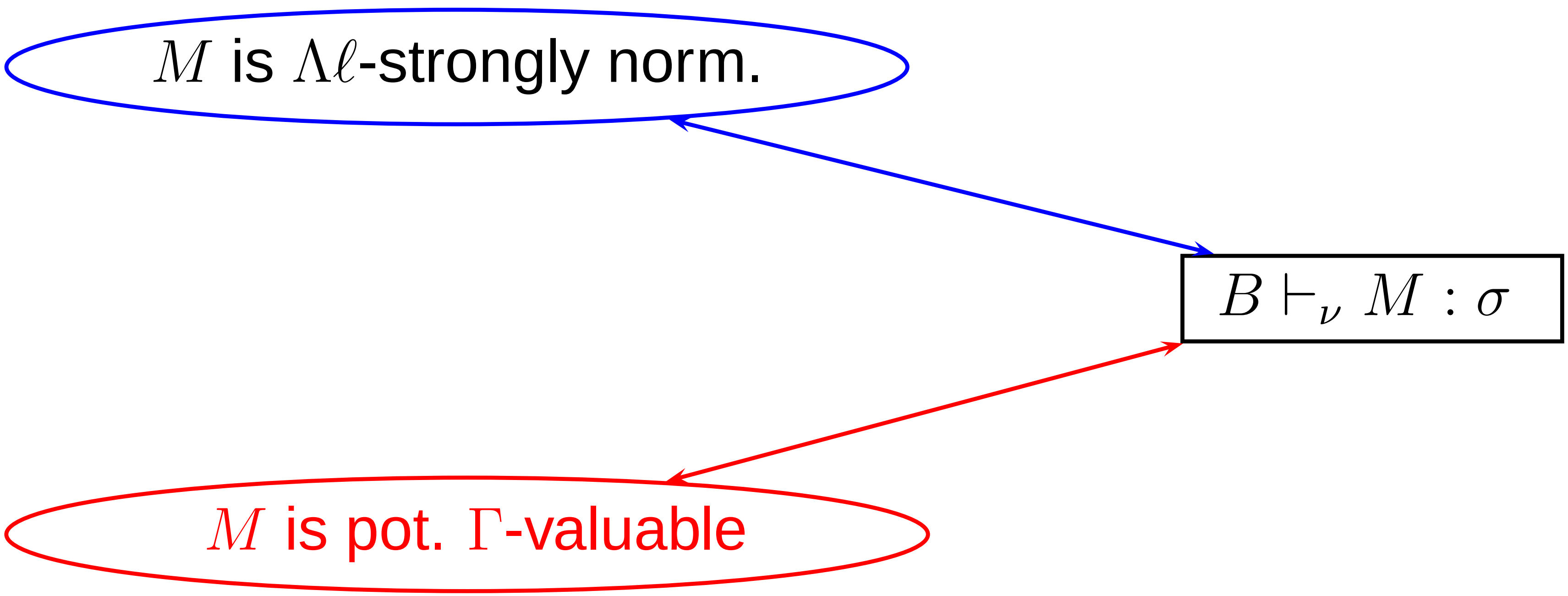

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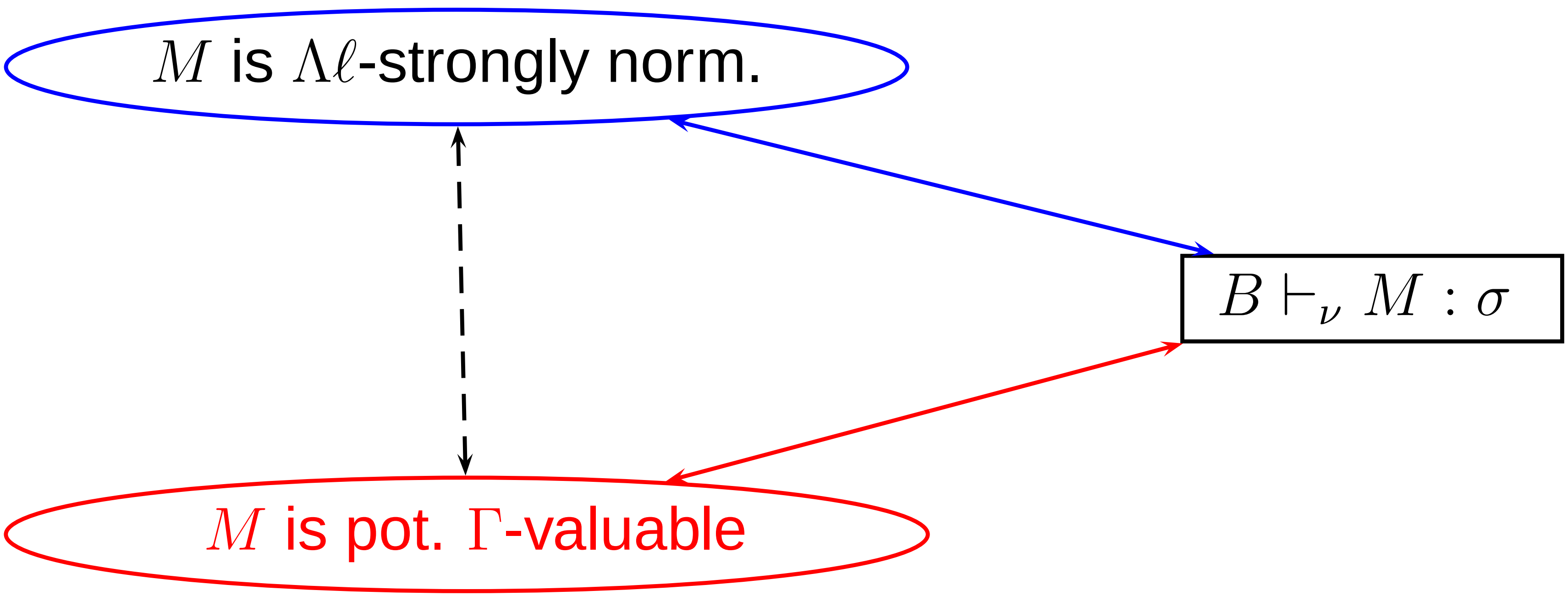

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A non lazy Call-By-Value

- Υ_i, Φ_i are defined as follows:

$$\Upsilon_0 = \text{Var} \qquad\qquad \Phi_i = \text{Var} \cup (\Upsilon_i)^0$$

$$\begin{aligned}\Upsilon_{i+1} = \text{Var} &\cup \{xM_1...M_n \mid M_k \in \Upsilon_i (1 \leq k \leq n)\} \\ &\cup \{\lambda \vec{x}.M \mid M \in \Upsilon_i\} \\ &\cup \{(\lambda x.P)QM_1...M_n \mid Q \in \Upsilon_i - (\Lambda^0 \cup \text{Var}), \\ &\qquad\qquad P[Q/x]M_1 \ldots M_n \rightarrow^*_{\Phi_i} R \in \Upsilon_i\}\end{aligned}$$

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- Let $\Upsilon = \cup_i \Upsilon_i$ and $\Phi = \mathrm{Var} \cup (\Upsilon)^0$

Some Properties of $\lambda\Phi$-calculus

M is Λ-strongly norm. $\longleftrightarrow$ $B \vdash M : \sigma$

M is Φ-solvable

M is pot. Φ-valuable

$M \to_{\Phi}^{*} N \in \Upsilon$

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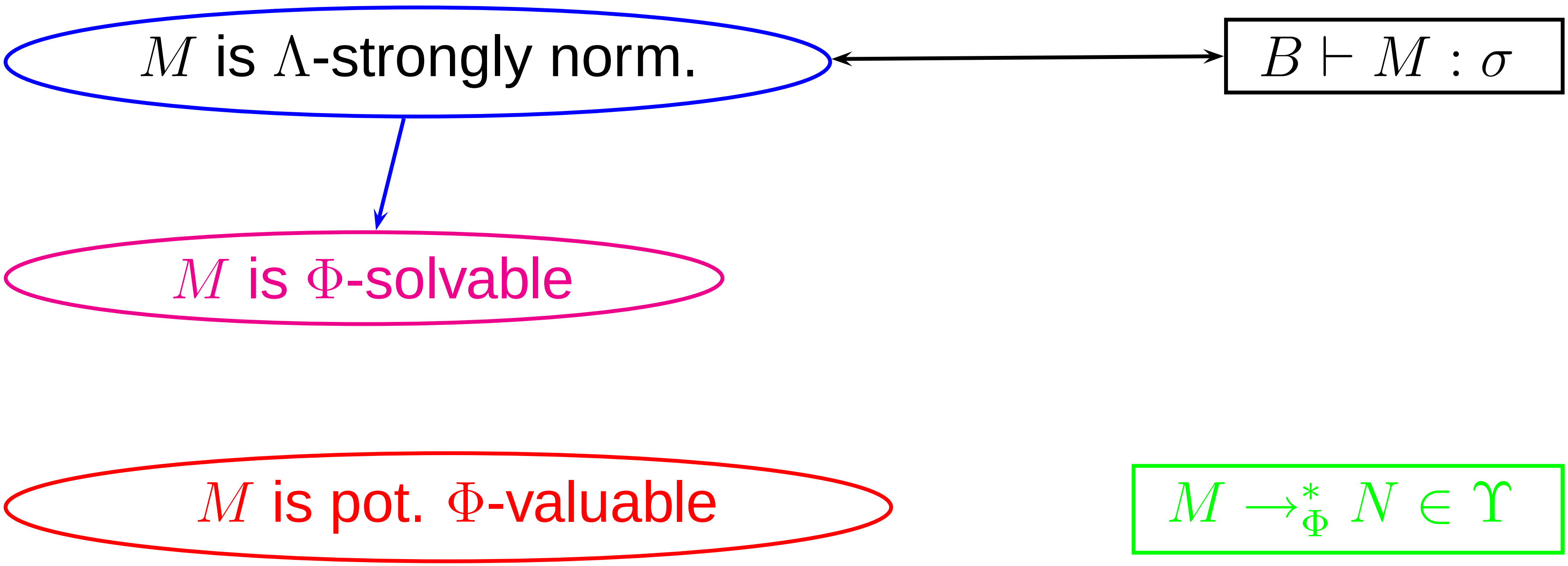

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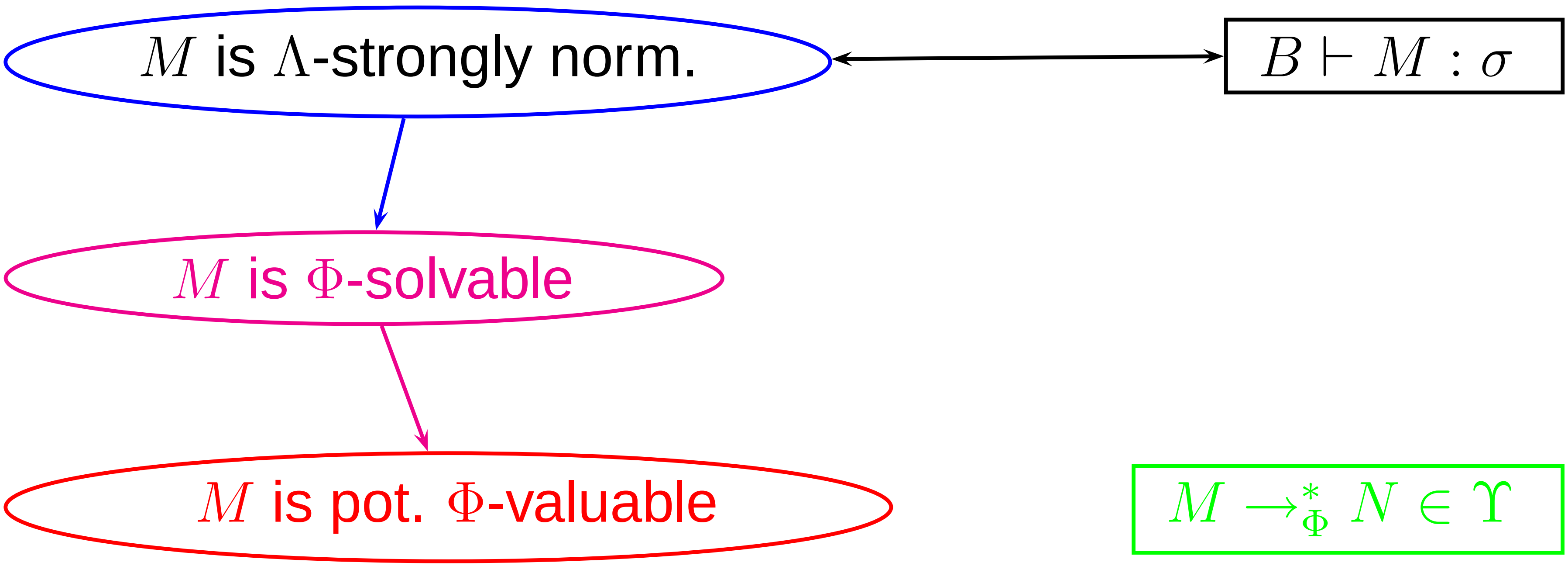

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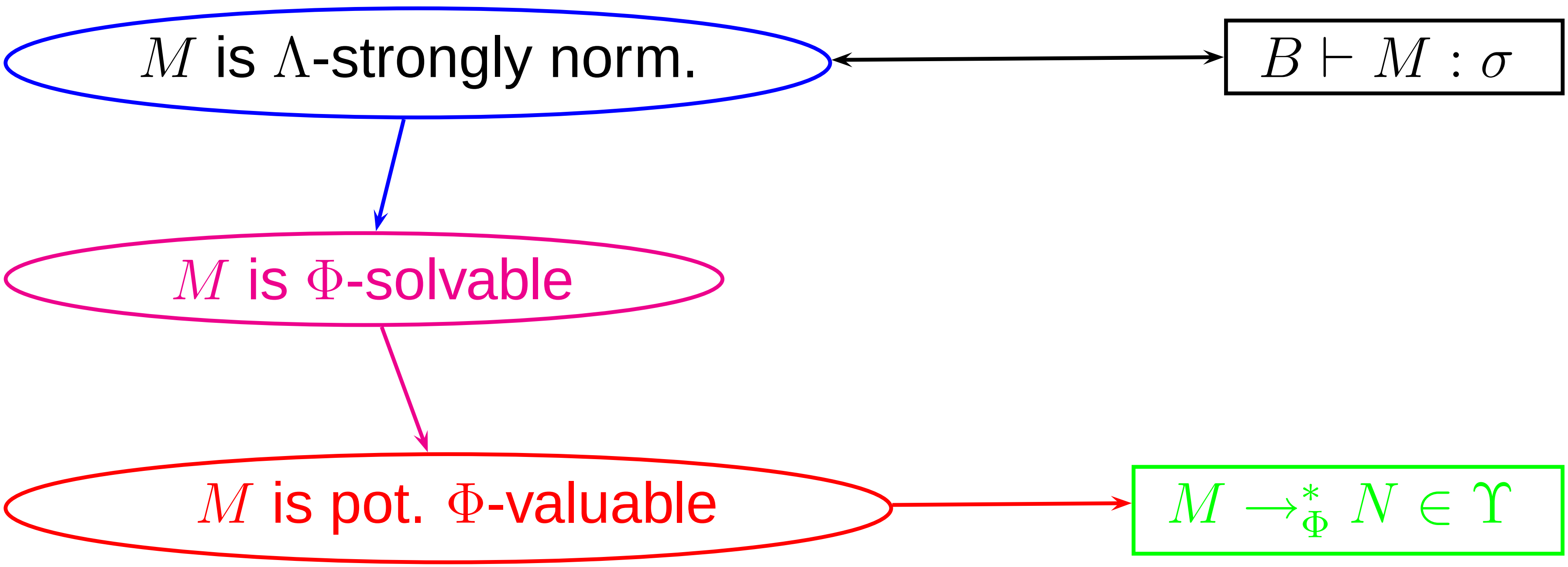

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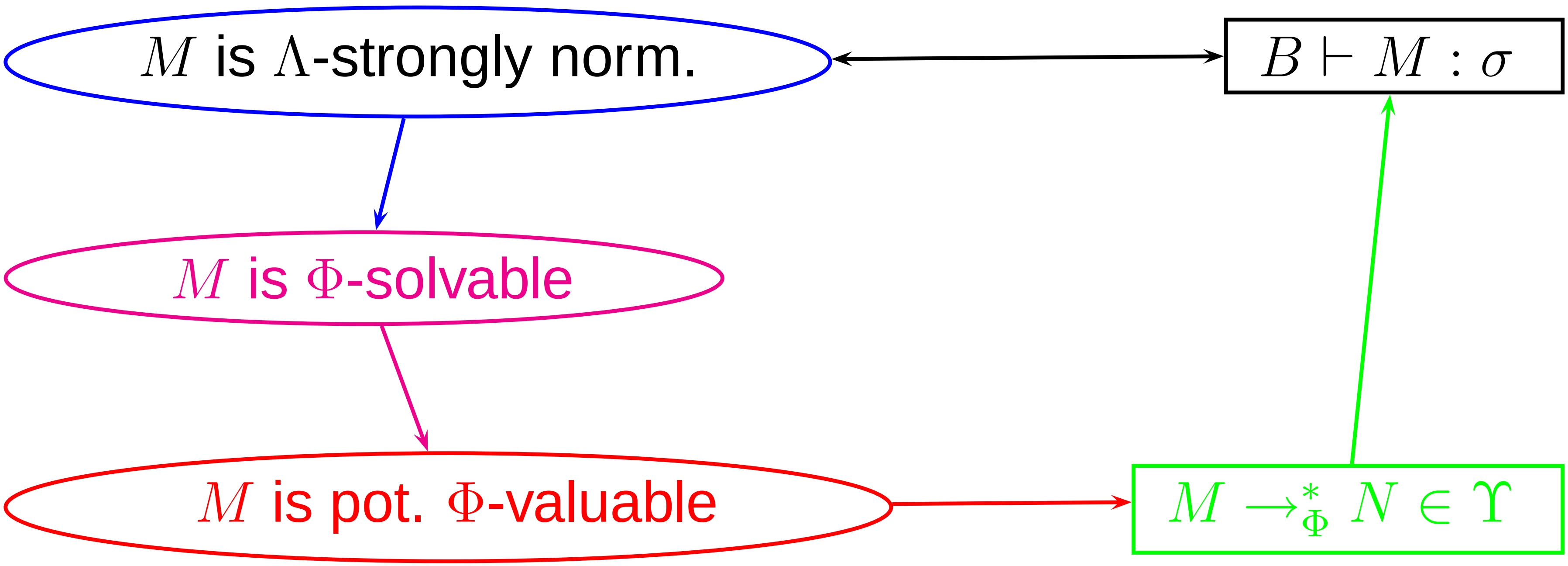

$\lambda\Delta$-model

$\lambda\Delta$-model is a quadruple $< \mathbb{D}, \mathbb{I}, \circ, [\![.]\!] >$ where

- $\mathbb{D}$ is a set
- $\circ$ is an operation on $\mathbb{D}$
- $\mathbb{I}$ is a subset of $\mathbb{D}$
- if $\rho : \mathrm{Var} \rightarrow \mathbb{I}$ then $[\![.]\!]_\rho$ maps terms in $\mathbb{D}$

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Intersection Types

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The set $T(C)$ of types is inductively defined as follows:

$$\sigma ::= \alpha \mid (\sigma \rightarrow \tau) \mid (\sigma \wedge \tau)$$

where $\alpha \in C$.

Intersection Type Theory

A intersection relation $\leq$ is a preorder relation on $T(C)$, closed under the following rules:

$$\frac{}{\sigma \leq \omega}\ (a) \qquad \frac{}{\sigma \leq \sigma}\ (r) \qquad \frac{\sigma \leq \rho,\ \rho \leq \tau}{\sigma \leq \rho}\ (t)$$

$$\frac{}{\sigma \leq \sigma \wedge \sigma}\ (b) \qquad \frac{}{\sigma \wedge \tau \leq \sigma}\ (c) \qquad \frac{}{\sigma \wedge \tau \leq \tau}\ (c')$$

$$\frac{}{(\sigma \to \tau) \wedge (\sigma \to \pi) \leq \sigma \to (\tau \wedge \pi)}\ (d) \qquad \frac{\sigma \leq \sigma',\ \tau \leq \tau'}{\sigma \wedge \tau \leq \sigma' \wedge \tau'}\ (e)$$

$$\frac{\sigma' \leq \sigma,\ \tau \leq \tau'}{\sigma \to \tau \leq \sigma' \to \tau'}\ (f) \qquad \frac{}{\sigma \to \omega \leq \omega \to \omega}\ (g)$$

$\leq$ induce a type theory $\simeq$.

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  - $\sigma \in I(C)$ and $\tau \notin I(C)$ imply $\sigma \leq_\nabla \tau$.

Intersection Type Assignment System

Let ∇ be a type system

$$\frac{}{B[\sigma/x] \vdash_\nabla x : \sigma}\ (var) \qquad \frac{}{B \vdash_\nabla M : \omega}\ (\omega)$$

$$\frac{B[\sigma/x] \vdash_\nabla M : \tau}{B \vdash_\nabla \lambda x.M : \sigma \to \tau}\ (\to I) \qquad \frac{\sigma \in I(C) \quad \begin{array}{c} B \vdash_\nabla M : \sigma \to \tau \\ B \vdash_\nabla N : \sigma \end{array}}{B \vdash_\nabla MN : \tau}\ (\to E)$$

$$\frac{B \vdash_\nabla M : \sigma \quad B \vdash_\nabla M : \tau}{B \vdash_\nabla M : \sigma \wedge \tau}\ (\wedge I) \qquad \frac{B \vdash_\nabla M : \sigma \quad \sigma \leq_\nabla \tau}{B \vdash_\nabla M : \tau}\ (\leq_\nabla)$$

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$$[\![M]\!]_{\rho}^{\mathcal{F}(\nabla)} = \{\sigma \in T(C) \mid \exists B \propto \rho \text{ such that } B \vdash_{\nabla} M : \sigma\}$$

where $B \propto \rho$ when $B(z) \in \rho(z)$, for all z.

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∇ is legal whenever, for all $\sigma \in I(C)$ and $\tau \not\leq_\nabla \omega$:

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implies $\exists \{i_1, ..., i_k\} \subseteq \{1, ..., n\}$ s.t.

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Filters models(Coppo-Dezani)

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- Full abstraction: $\sqsubseteq_{\mathcal{F}} = \preceq_{\mathbf{O}}$

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Lazy Evaluations (Plotkin 1975)

Call by name (said simply lazy)

$$\frac{}{\lambda x.M \Downarrow_{\mathbf{L}} \lambda x.M}\ (lazy) \qquad \frac{P[Q/x]M_1 \ldots M_m \Downarrow_{\mathbf{L}} N}{(\lambda x.P)QM_1 \ldots M_m \Downarrow_{\mathbf{L}} N}\ (head)$$

Call by value

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$$\frac{Q \Downarrow_{\mathbf{V}} Q' \qquad P[Q'/x]M_1 \ldots M_m \Downarrow_{\mathbf{V}} N}{(\lambda x.P)QM_1 \ldots M_m \Downarrow_{\mathbf{V}} N}\ (head)$$

Operational Semantics

Let $M, N \in \Lambda^0$

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$M \approx_{\mathbf{O}} N$ if and only if $M \preceq_{\mathbf{O}} N$ and $N \preceq_{\mathbf{O}} M$.

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- $\angle$ is the type system $< C_\angle, \leq_\angle, I(C_\angle) >$ where $\leq_\angle$ is the least intersection relation, $C_\angle = \{\omega\}$ and $I(C_\angle) = T(C_\angle)$.

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$\mathcal{L}$-Characterizations

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- $\sigma \simeq_{\angle} \omega$ if and only if $\sigma \equiv \underbrace{\omega \wedge \ldots\ldots \wedge \omega}_{n}$.

$\mathcal{V}$-Characterizations

- $B \vdash_{\surd} M : \omega \to \omega$ if and only if $M \Downarrow_{\mathbf{V}}$.
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  $\forall i \leq n$, $\sigma_i \simeq_{\surd} \tau_1^i \to \ldots \to \tau_{m_i}^i \to \omega \to \omega$.
- $\sigma \simeq_{\surd} \omega$ if and only if $\sigma \equiv \underbrace{\omega \wedge \ldots\ldots \wedge \omega}_{n}$.

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$M \trianglelefteq^{\mathcal{L}} N$ if and only if $M \trianglelefteq_{\sigma}^{\mathcal{L}} N$, for all σ.

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$\trianglelefteq_\sigma^V$ is a relation on Λ^0 defined as follows:

- $M \trianglelefteq_\omega^V N$ is true;
- $M \trianglelefteq_{\sigma\to\tau}^V N$ where $\tau \simeq_{\surd} \omega$, if and only if
  $B \vdash_{\surd} M : \omega \to \omega$ implies $B \vdash_{\surd} N : \omega \to \omega$;
- $M \trianglelefteq_{\sigma\to\tau}^V N$ where $\tau \not\simeq_{\surd} \omega$, if and only if
  $P \in \Gamma^0$-val. and $B \vdash_{\surd} P : \sigma$ imply $MP \trianglelefteq_\tau^V NP$;
- $M \trianglelefteq_{\sigma\wedge\tau}^V N$ if and only if $M \trianglelefteq_\sigma^V N$ and $M \trianglelefteq_\tau^V N$.

$M \trianglelefteq^V N$ if and only if $M \trianglelefteq_\sigma^V N$, for all σ.

$\unlhd^{\mathcal{L}}$-Properties

Let $M, N \in \Lambda^0$.

- $\unlhd^{\mathcal{L}}$ is a preorder relation

$\trianglelefteq^{\mathcal{L}}$-Properties

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- $M \sqsubseteq_{\mathcal{L}} N$ implies $M \trianglelefteq^{\mathcal{L}} N$

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$M \triangleq N$ if and only if $M \trianglelefteq^{\mathcal{L}} N$ and $N \trianglelefteq^{\mathcal{L}} M$.

$\trianglelefteq^V$-Properties

Let $M, N \in \Lambda^0$.

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- $M \trianglelefteq^V N$ if and only if $M \preceq_{\mathbf{V}} N$

$M \triangleq N$ if and only if $M \trianglelefteq^V N$ and $N \trianglelefteq^V M$.

$\mathcal{L}\mathcal{L}$-model

$\triangleq$ induces a equiv. relation on $\mathcal{F}^0(\angle)$, the set of filters being interpretations of closed terms.

- $[f]$ will be the equivalence class of f w.r.t. $\triangleq$, while $\mathcal{F}^0_{\triangleq}$ will be the set of all such classes.

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- $\mathcal{I}^0_{\triangleq} = \{[f] \in \mathcal{F}^0_{\triangleq} \mid \exists M \in \Lambda^0$ s.t. $[\![M]\!]^{\mathcal{F}(\angle)}_{\rho} = f\}$.

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- $[\![M]\!]^{\mathcal{LL}}_{\zeta} = [[\![M]\!]^{\mathcal{F}(\angle)}_{\rho}]$, where $\rho(x) \in \zeta(x)$.

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- $[\![M]\!]^{\mathcal{LL}}_{\zeta} = [[\![M]\!]^{\mathcal{F}(\angle)}_{\rho}]$, where $\rho(x) \in \zeta(x)$.
- Let $\mathcal{LL}$ be the quadruple: $< \mathcal{F}^0_{\triangleq}, \mathcal{I}^0_{\triangleq}, \circ, [\![.]\!]^{\mathcal{LL}} >$.

$\mathcal{VV}$-model

$\triangleq$ induces a equiv. relation on $\mathcal{F}^0(\surd)$, the set of filters being interpretations of closed terms.

- $[f]$ will be the equivalence class of f w.r.t. $\triangleq$, while $\mathcal{F}^0_{\triangleq}$ will be the set of all such classes.
- $\mathcal{I}^0_{\triangleq} = \{[f] \in \mathcal{F}^0_{\triangleq} \mid \exists M \in \Gamma^0$ s.t. $[\![M]\!]^{\mathcal{F}(\surd)}_{\rho} = f\}$.
- $[f] \circ_{\triangleq} [g] = [f \circ_{\surd} g]$, for all $[f], [g] \in \mathcal{F}^0_{\triangleq}$.
- $[\![M]\!]^{\mathcal{VV}}_{\zeta} = [[\![M]\!]^{\mathcal{F}(\surd)}_{\rho}]$, where $\rho(x) \in \zeta(x)$.
- Let $\mathcal{VV}$ be the quadruple: $< \mathcal{F}^0_{\triangleq}, \mathcal{I}^0_{\triangleq}, \circ, [\![.]\!]^{\mathcal{VV}} >$.

Full Abstraction

- $\mathcal{LL}$ is a $\lambda\Lambda$-model.
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Full Abstraction

- $\mathcal{LL}$ is a $\lambda\Lambda$-model.
- $\mathcal{VV}$ is a $\lambda\Gamma$-model.
- The model $\mathcal{LL}$ is fully abstract with respect to the **L**-operational semantics.
- The model $\mathcal{VV}$ is fully abstract with respect to the **V**-operational semantics.