Type theory via category theory

Ruggero Pagnan
Dipartimento di Matematica dell'Università
degli studi di Genova

STT $x{:}\,\sigma$ $(\sigma{:}\,Type \quad \rightarrow,\ \times,\ 1,\ +,\ 0)$

DTT $x{:}\,\sigma$ $\tau(x){:}\,Type$

PTT $x{:}\,\sigma$ $\alpha{:}\,Type$ $\tau(x, \alpha){:}\,Type$

Polymorphic type theory

$\lambda \rightarrow$

$$\lambda x{:}\,\alpha.x{:}\,\alpha \rightarrow \alpha$$

$$\lambda x{:}\,\sigma.x{:}\,\sigma \rightarrow \sigma$$

$\lambda 2$

$$I = \lambda \alpha{:}\,Type.\lambda x{:}\,\alpha.x{:}\,\Pi\alpha{:}\,Type.(\alpha \rightarrow \alpha)$$

$$I\sigma = \lambda x{:}\,\sigma.x{:}\,\sigma \rightarrow \sigma$$

$\lambda\omega$

$\lambda\alpha{:}\,Type \rightarrow Type.\lambda\beta{:}\,Type.\alpha\,\beta \rightarrow \beta{:}\,(Type \rightarrow Type) \rightarrow (Type \rightarrow Type)$

PTT $\quad \sigma\colon Type \quad M\colon \sigma \quad A\colon Kind \quad M\colon Kind$

$$\Gamma = (x_1\colon \sigma_1, \ldots, x_m\colon \sigma_m)$$

$$\Xi = (\alpha_1\colon A_1, \ldots, \alpha_n\colon A_n)$$

$$\alpha_1\colon A_1, \ldots, \alpha_n\colon A_n | x_1\colon \sigma_1, \ldots, x_m\colon \sigma_m \vdash M\colon \tau$$

Many typed:

$$\Sigma = (T, \mathcal{F}) \quad |\Sigma| = T \quad \mathcal{F}: T^* \times T \longrightarrow \mathbf{Sets}$$

$$\mathcal{F}(\langle \sigma_1, \ldots, \sigma_n \rangle, \sigma_{n+1}) \ni F: \sigma_1, \ldots, \sigma_n \to \sigma_{n+1}$$

Higher order: $Prop: 1 \longrightarrow T = |\Sigma|$

Polymorphic:

$$\Sigma \qquad |\Sigma| = K \qquad Type: 1 \longrightarrow K$$

$$(\Sigma_a)_{a \in K^*} \qquad K^* \ni a = \langle A_1, \ldots, A_n \rangle$$

$$|\Sigma_a| \ni \alpha_1: A_1, \ldots, \alpha_n: A_n \vdash \sigma: Type$$

Set theoretic model of $\langle \Sigma, (\Sigma_a) \rangle$

- $$[\![Type]\!] = \mathcal{U}$$

$$\alpha_1 : A_1, \ldots, \alpha_n : A_n \vdash \sigma : Type$$

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{[\![\sigma]\!]} \mathcal{U}$$

- $$F : \sigma_1, \ldots, \sigma_m \longrightarrow \sigma_{m+1} \qquad \Sigma_{(A_1, \ldots, A_n)}$$

$$[\![\sigma]\!](\vec{a}) \times \cdots \times [\![\sigma]\!](\vec{a}) \xrightarrow{[\![F]\!](\vec{a})} [\![\sigma_{m+1}]\!](\vec{a})$$

$$\vec{a} \in [\![A_1]\!] \times \cdots \times [\![A_n]\!]$$

Fibred categories: indexing

pointwise $(X_i)_{i \in I}$ $I \to \mathbf{Sets}$

display $\begin{array}{c} X \\ \downarrow{\scriptstyle \psi} \\ I \end{array}$ $\psi^{-1}(i) = \{x \in X | \psi(x) = i\}$

Example:

$$\begin{array}{c} I \times X \\ \downarrow{\scriptstyle \pi} \\ I \end{array}$$ constant family

$\mathbf{Sets}/_I$

$$\begin{array}{c} X \\ \Big\downarrow{\scriptstyle \psi} \\ I \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \swarrow & \\ & I & \end{array} \qquad f = (\, X_i \xrightarrow{f_i} Y_i \,)_{i \in I}$$

$\mathbf{Sets}^{\rightarrow}$

$$\begin{array}{c} X \\ \Big\downarrow{\scriptstyle \psi} \\ I \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \Big\downarrow & & \Big\downarrow \\ I & \xrightarrow[\quad u \quad]{} & J \end{array} \qquad f = (\, X_i \xrightarrow{f_i} Y_{u(i)} \,)_{i \in I}$$

$cod\colon \mathbf{Sets}^{\rightarrow} \rightarrow \mathbf{Sets}$

Fibred categories: substitution

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \psi} \\ I & \xrightarrow[u]{} & J \end{array} \qquad (Y_j)_{j \in J}$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u^*(\psi)}\downarrow & & \downarrow{\scriptstyle \psi} \\ I & \xrightarrow[u]{} & J \end{array} \qquad (X_i = Y_{u(i)})_{i \in I}$$

$$X = \{(i, y) \in I \times Y | u(i) = \psi(y)\}$$

Weakenig:

$$
\begin{array}{ccc}
Y \times I & \xrightarrow{\pi} & Y \\
{\scriptstyle \psi \times id}\downarrow & & \downarrow{\scriptstyle \psi} \\
J \times I & \xrightarrow[\pi]{} & J
\end{array}
$$

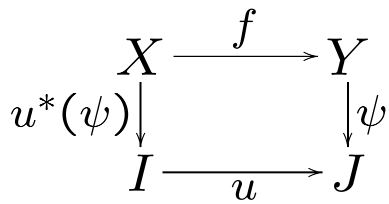
$$(\psi \times id)^{-1}(j,i) \simeq \psi^{-1}(j) \times I$$

Contraction:

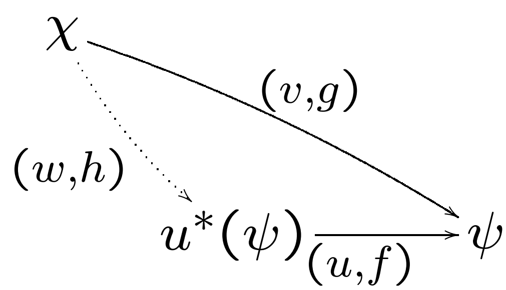
$$
\begin{array}{ccc}
\bullet & \longrightarrow & Y \\
{\scriptstyle \delta^*(\psi)}\downarrow & & \downarrow{\scriptstyle \psi} \\
J & \xrightarrow[\delta]{} & J \times J
\end{array}
$$

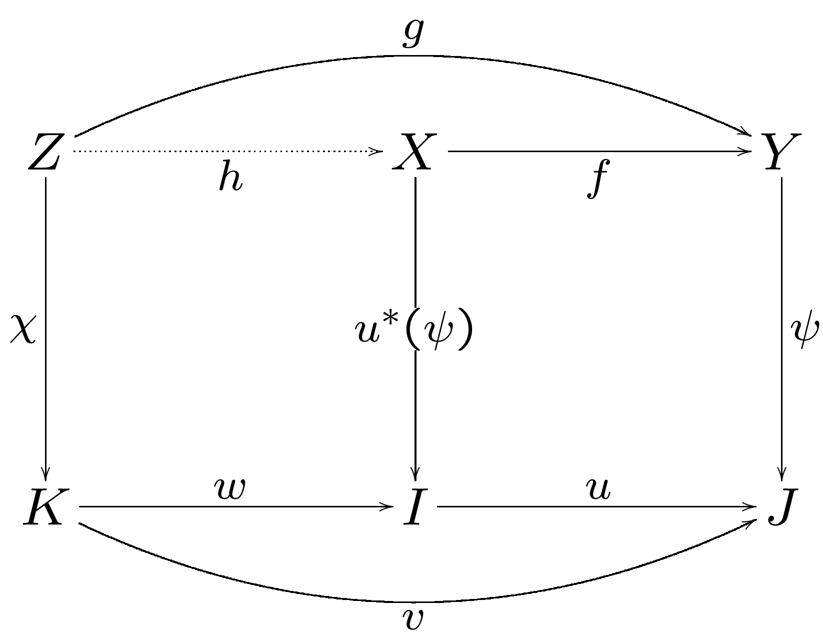
$$\delta^*(\psi)_j \simeq \{y \in Y | \psi(y) = (j,j)\} = Y_{(j,j)}$$

$$cod\colon \mathbf{Sets}^{\to} \longrightarrow \mathbf{Sets}$$

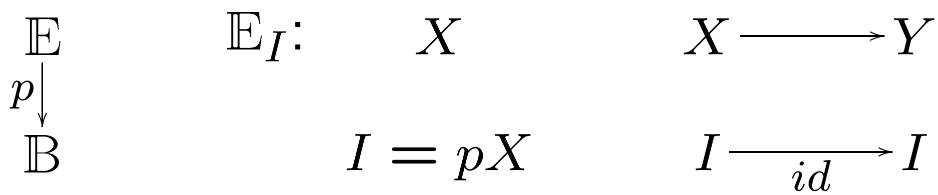


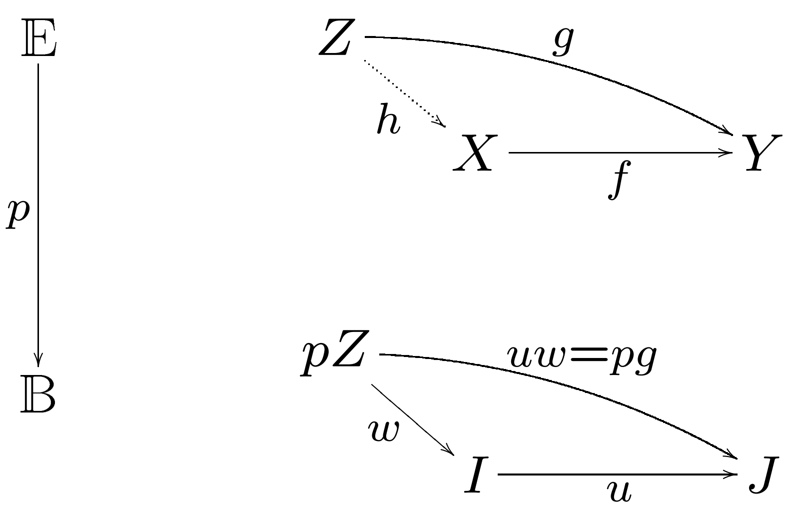
$$u^*(\psi) \longrightarrow \psi$$





Fibrations





Polymorphic fibrations

$$\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & T \\ & & \\ pX & \underset{u}{\dashrightarrow} & pT \end{array}$$

$$\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array} \qquad \begin{array}{ccc} X = (i\colon I \vdash X_i\colon Type) & \xrightarrow{f} & T \\ & & \\ I & \underset{u}{\dashrightarrow} & Type \end{array}$$

ω-**Sets**

$$(X, E) \qquad \forall\, x \in X. E(x) \subseteq \mathbb{N}$$

$$(X, E) \xrightarrow{f} (Y, E) \qquad X \xrightarrow{f} Y$$

$$\forall\, x \in X. \forall\, n \in E(x). \exists\, e \in \mathbb{N}. e \cdot n \downarrow . e \cdot n \in E(f(x))$$

PERs

$R \subseteq \mathbb{N} \times \mathbb{N}$ symmetric and transitive

$|R| = \{n \in \mathbb{N} | nRn\}$ domain

$[n] = [n]_R = \{m \in \mathbb{N} | mRn\}$

$\mathbb{N}/R = |R|/R = \{[n] | n \in |R|\}$ quotient

$PER = \{R \subseteq \mathbb{N} \times \mathbb{N} | R \textit{ is a } PER\}$

PER

$R \in PER$

$R \to S \qquad f \colon \mathbb{N}/R \to \mathbb{N}/S$

$\exists\, e \in \mathbb{N}. \forall\, n \in |R|. f([n]_R) = [e \cdot n]_S$

UFam(**PER**)

$$(R_i)_{(I,E)} \qquad R_i \in PER \qquad (I,E) \in \omega - \textbf{Sets}$$

$$(R_i)_{i\in(I,E)} \xrightarrow{(u,f)} (S_j)_{j\in(J,E)}$$

$$u\colon (I,E) \longrightarrow (Y,E)$$

$$f = (\, f_i\colon R_i \longrightarrow S_{u(i)}\,)_{i\in I}$$

$\exists\, e \in \mathbb{N}. \forall\, i \in I. \forall\, n \in E(i). e \cdot n$ tracks f_i in **PER**

$$
\begin{array}{ccc}
UFam(\mathbf{PER}) & & (R_i)_{i \in (I,E)} \\
\downarrow & & \downarrow \\
\omega - \mathbf{Sets} & & (I, E)
\end{array}
$$

Simple fibrations

$s(\mathbb{B})$

$(I, X) \in \mathbb{B} \times \mathbb{B}$

$(I, X) \xrightarrow{(u,f)} (J, Y) \qquad I \xrightarrow{u} J \qquad f \colon I \times X \longrightarrow Y$

$$\begin{array}{ccc} s(\mathbb{B}) & & (I, X) \\ \downarrow & & \downarrow \\ \mathbb{B} & & I \end{array}$$

$s(\mathbb{B})_I$

$X \in \mathbb{B} \qquad I \times X \longrightarrow Y$

CT-structures

$(\mathbb{B}, T) \qquad \emptyset \neq T \subseteq Obj(\mathbb{B})$

$s(T)$

$(I, X) \in \mathbb{B} \times T$

$(I, X) \xrightarrow{(u,f)} (J, Y) \qquad I \xrightarrow{u} J \qquad f\colon I \times X \longrightarrow Y$

$T = \mathbb{B} \qquad T = \{\Omega\}$

Display map categories

$(\mathbb{B}, D) \qquad \emptyset \neq D \subseteq Arr(\mathbb{B})$

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle D \ni u^*(\psi)}\big\downarrow & & \big\downarrow{\scriptstyle \psi \in D} \\ J & \xrightarrow[u]{} & I \end{array}$$

$$\begin{array}{c} D^{\rightarrow} \\ \big\downarrow{\scriptstyle cod} \\ \mathbb{B} \end{array} \qquad\qquad D/_{I}$$

$$(\mathbb{B}, T) \quad \emptyset \neq T \subseteq Obj(\mathbb{B})$$

$$\begin{array}{c} I \times X \\ \big\downarrow{\scriptstyle \pi} \\ I \end{array}$$