Context Semantics and Implicit Computational Complexity

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Outline

Logical Characterization of Complexity Classes

- Various branches of Mathematical Logic have been touched by Implicit Computational Complexity:
  - **Recursion Theory**;
  - **Proof Theory** (Curry-Howard correspondence);
  - **Model Theory** (Finite Model Theory).
- We are mainly interested in proof-theoretical and recursion-theoretical characterizations.
  - They induce programming languages with bounded expressive power.
  - Resource-aware type-systems for existing programming languages can be designed from some of these systems.

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The Need for a Unifying Framework

- Many systems are sound and complete w.r.t. relevant complexity classes (polytime functions, elementary functions, etc.).
- Proof techniques for soundness results are ad-hoc and cannot be easily generalized.
- As a consequence:
  - There are many **open problems**. For example: the impact of linearity costraint on the expressive power of higher-order recursion [Hofmann97,BNS00] has not been analyzed precisely.
  - It is not known whether **combining different systems** characterizing the same class would break soundness.
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Context Semantics

- *Intensional* ways of giving semantics to logics and programming languages:
  - **Game semantics** [AJM92];
  - **Geometry of interaction** [Girard89]
  - **Context semantics** [GAL92a,GAL92b].
- Game semantics and geometry of interaction have been already used to study quantitative properties of programs and proofs [BaillotPedicini01,Ghica05].
- We chose context semantics because of its simplicity.
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Characterizations of Complexity Classes by Higher-Order Recursion

- We obtain characterizations of the following complexity classes:
  - PR, the set of (first-order) primitive recursive functions.
  - E, the class of elementary time computable functions.
  - P, the class of polynomial time computable functions.
- Some of them are already known, while other ones are novel.
- The above characterizations are obtained from higher-order (primitive) recursion by fine-tuning two parameters, namely:
  - The class of "contractible" types.
  - Ramification constraints.
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Context Semantics and Linear Logic

From the context semantics of any linear logic proof-net G, a *weight* $W_G \in \mathbb{N} \cup \{\omega\}$ can be defined in such a way that:

Theorem

There is a polynomial $p : \mathbb{N}^2 \to \mathbb{N}$ such that for every proof-net G, G normalizes in at most $p(W_G, |G|)$ steps and the size of any reduct H of G is at most $p(W_G, |G|)$.

Theorem

Let G be a proof-net. If $W_G = \omega$, then G diverges, otherwise there is H with $G \to^{W_G} H$.

Terms

- Let us first consider the sub-calculus handling the algebra $\mathbb{U}$ of unary integers.
- Terms:

$$M ::= x \mid 0 \mid s \mid MM \mid \lambda x.M \mid M \{\!\{M, M\}\!\} \mid M \langle\!\langle M, M\rangle\!\rangle$$

- Reduction rules are call-by-value:

$$(\lambda x.M)\, V \to M\{V/x\}$$

$$0 \langle\!\langle M, N\rangle\!\rangle \to N \qquad 0 \{\!\{M, N\}\!\} \to N$$

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Types

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$$A ::= \mathbb{U}^n \mid A \multimap A$$

- Type-assignment rules:

$$\frac{}{\Gamma, x : A \vdash x : A}\ A \qquad \frac{\Gamma, x : A, y : A \vdash M : B}{\Gamma, z : A \vdash M\{z/x, z/y\} : B}\ C$$

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Accomodating Arbitrary Algebras

- The set of base types can be extended to arbitrary free algebras. Examples are
  - **Binary Strings**: Constructors are $\varepsilon, 0, 1$ and have arity $0, 1, 1$, respectively.
  - **Binary Trees**: Constructors are $leaf, node$ and have arity $0, 2$, respectively.
  - **Binary Trees with binary labels**: Constructors are $leaf, nodezero, nodeone$ and have arity $0, 2, 2$, respectively.
- The first one is a word algebra. The other ones are tree algebras.
- $\mathbf{W}$ is the class of word algebras, $\mathbf{A}$ is the class of tree algebras, $\mathbf{T}$ is the class of all types.

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Subsystems

- Expressive power of the whole calculus: too large!.
- We are interested in two different constraints on the type-system:
  - We can allow **contraction** to be applicable only to some types (in the style of linear logic [Girard87]).
  - We can introduce **ramification** (similarly to [Hofmann97])
- In this way, we can define various subsystems. For example:
  - A subsystem with ramification where contraction is applicable only to word algebras: $\mathbf{RH}(\mathbf{W})$.
  - A subsystem without ramification where contraction cannot be applied: $\mathbf{H}(\emptyset)$.

  All these fragments enjoy subject-reduction.

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The Results

- Here are the obtained results:

	$\mathbf{T}$	$\mathbf{A}$	$\mathbf{W}$	$\emptyset$
$\mathbf{H}(\cdot)$	PA	PR	PR	PR
$\mathbf{RH}(\cdot)$	E	E	P	P

- Going from $\mathbf{RH}(\mathbf{W})$ to $\mathbf{RH}(\mathbf{A})$ greatly increases the expressive power.
- Going from $\mathbf{RH}(\emptyset)$ to $\mathbf{RH}(\mathbf{W})$ does not change anything.
- Forbidding contraction on higher-order types (almost always) "neutralizes" higher-order recursion.

Parameters Controlling Computing Time - I

- All the results we have cited are characterization results: both soundness and completeness hold.
- Two parameters control computing time:
  - The **recursion depth** $R(M)$ of a term M is the nesting depth of recursions inside M.
  - The **highest tier** $I(\pi)$ of a type derivation π is the maximum integer i such that there is an instance of $E^R_{\multimap}$ in π with the shape

$$\frac{\pi_1 \quad \dots \quad \pi_n \quad \Delta \vdash L : \mathbb{A}^i}{\Gamma, \Delta \vdash L \langle\langle M_1, \dots, M_n \rangle\rangle : C}$$

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Parameters Controlling Computing Time - II

- From the following, you can easily understand in which sense the recursion depth and the highest tier control the computing time:

Theorem (Polytime soundness for $\mathbf{RH(W)}$)

For every $n, m \in \mathbb{N}$, there are polynomials p_m^n such that for every type derivation $\pi : \Gamma \vdash_{\mathbf{RH(W)}} M : A$, the time needed to normalize M is at most then $p_{R(M)}^{I(\pi)}(|M|)$.

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A Crucial Result

- The algebraic potential size $A(M)$ of a term M is is the maximum natural number n such that $M \rightsquigarrow^* N$ and there is a redex in N whose argument is a free algebra term t with $|t| = n$.
- We get:

Theorem

For every $d \in \mathbb{N}$ there are polynomials $p_d, q_d : \mathbb{N}^2 \to \mathbb{N}$ such that whenever $\pi : \Gamma \vdash M : A$ and $M \rightsquigarrow^n N$, then $n \leq p_{R(M)}(|M|, A(M))$ and $|N| \leq q_{R(M)}(|M|, A(M))$

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Context Semantics in Five Minutes

- Type derivations as graphs. From a type derivation π we can obtaine a graph G_π.
- For every graph, a set of trees. From a graph G, a set of trees $\mathcal{T}(G)$.
- There is a morphism Φ extracting a free algebra term t from every tree in $\mathcal{T}(G)$.
- If $\pi : \Gamma \vdash M : A$, $M \rightsquigarrow^* N$ and t appear as an argument of a redex in N, then $t \in \Phi(\mathcal{T}(G_\pi))$.
- Interestingly:
  - The size of G_π is proportional to that of π.
  - The size of $\Phi(T)$ is proportional to that of T.
  - Studying the combinatorial behaviour of $\mathcal{T}(\cdot)$ as a map suffices!

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Trees in $\mathcal{T}(G)$

- Branches correspond to execution paths.
- Nodes are labelled with contexts

$$(A[\cdot], (u_1[\cdot], t_1) \ldots (u_n[\cdot], t_n))$$

  where:
  - $A[\cdot]$ is a type-context guiding tree costruction;
  - each $(u_i[\cdot], t_i)$ keeps track of which step function copy we are visiting. $u_i[\cdot]$ is a free-algebra-term-context, while t_i is a free algebra term.
- This is somehow reminiscent of token machines and game semantics.

An Example

- Consider $\mathbf{UnAdd} \equiv \lambda x.\lambda y.x\langle\langle \lambda w.\lambda z.c_1^{\mathbb{U}} z, y\rangle\rangle$.
  $\mathbf{UnAdd}\lceil n\rceil\lceil m\rceil \rightsquigarrow^* \lceil n+m\rceil$ for every $n, m \in \mathbb{N}$.
- Consider the following sketch of a type derivation for $\mathbf{UnAdd}\lceil 1\rceil\lceil 1\rceil$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{w:\mathbb{U}^1 \vdash c_1^{\mathbb{U}}} \quad \dfrac{}{z:\mathbb{U}^0 \vdash z:\mathbb{U}^0}}{w:\mathbb{U}^1, z:\mathbb{U}^0 \vdash c_1^{\mathbb{U}} z:\mathbb{U}^0}}{w:\mathbb{U}^1 \vdash \lambda z.c_1^{\mathbb{U}} z:\mathbb{U}^0 \multimap \mathbb{U}^0}}{\vdash \lambda w.\lambda z.c_1^{\mathbb{U}} z:\mathbb{U}^1 \multimap \mathbb{U}^0 \multimap \mathbb{U}^0} \quad \dfrac{}{y:\mathbb{U}^0 \vdash y:\mathbb{U}^0} \quad \dfrac{}{x_{\mathbb{U}}^1 \vdash x:\mathbb{U}^1}}{x:\mathbb{U}^1, y:\mathbb{U}^0 \vdash x\langle\langle \lambda w.\lambda z.c_1^{\mathbb{U}} z, y\rangle\rangle:\mathbb{U}^0}}{\vdash \mathbf{UnAdd}:\mathbb{U}^1 \multimap \mathbb{U}^0 \multimap \mathbb{U}^0} \qquad \dfrac{}{\vdash \lceil 1\rceil:\mathbb{U}^1} \quad \dfrac{}{\vdash \lceil 1\rceil:\mathbb{U}^0}}{\vdash \mathbf{UnAdd}\lceil 1\rceil\lceil 1\rceil:\mathbb{U}^0}$$

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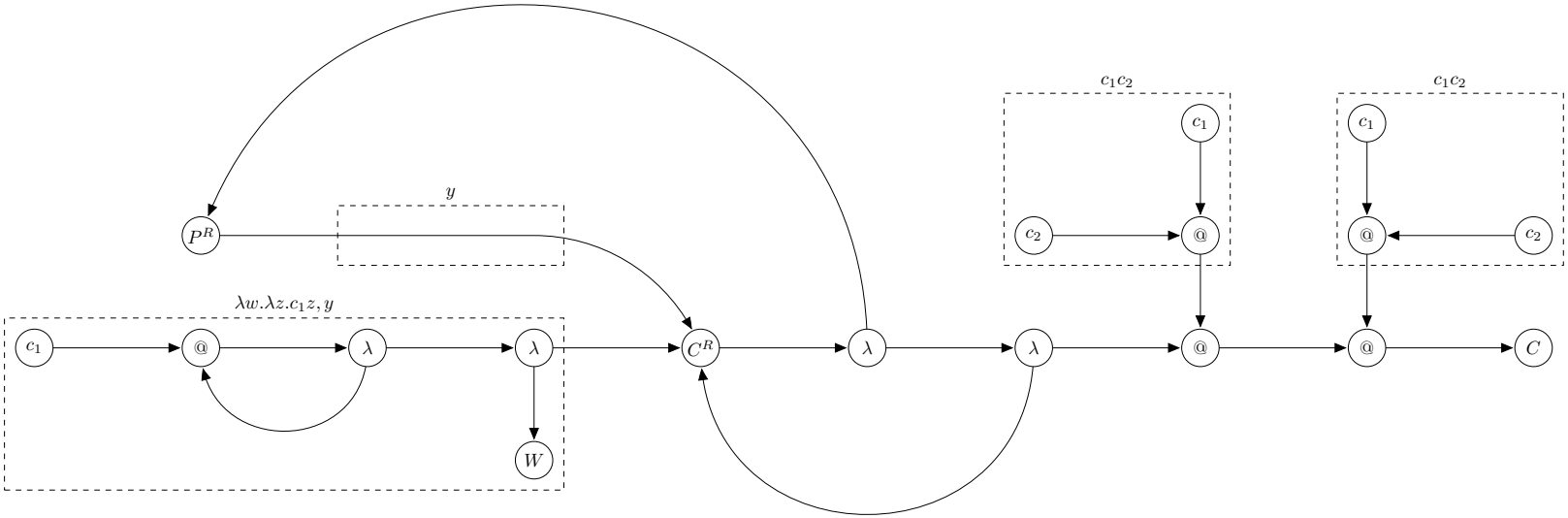

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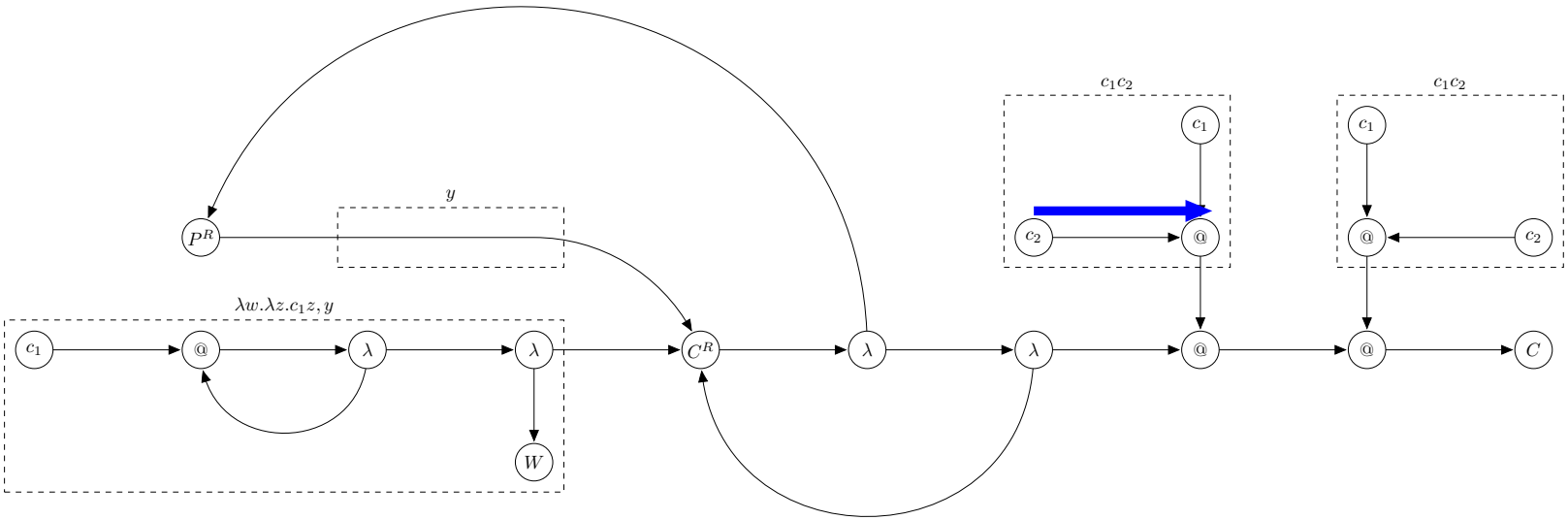

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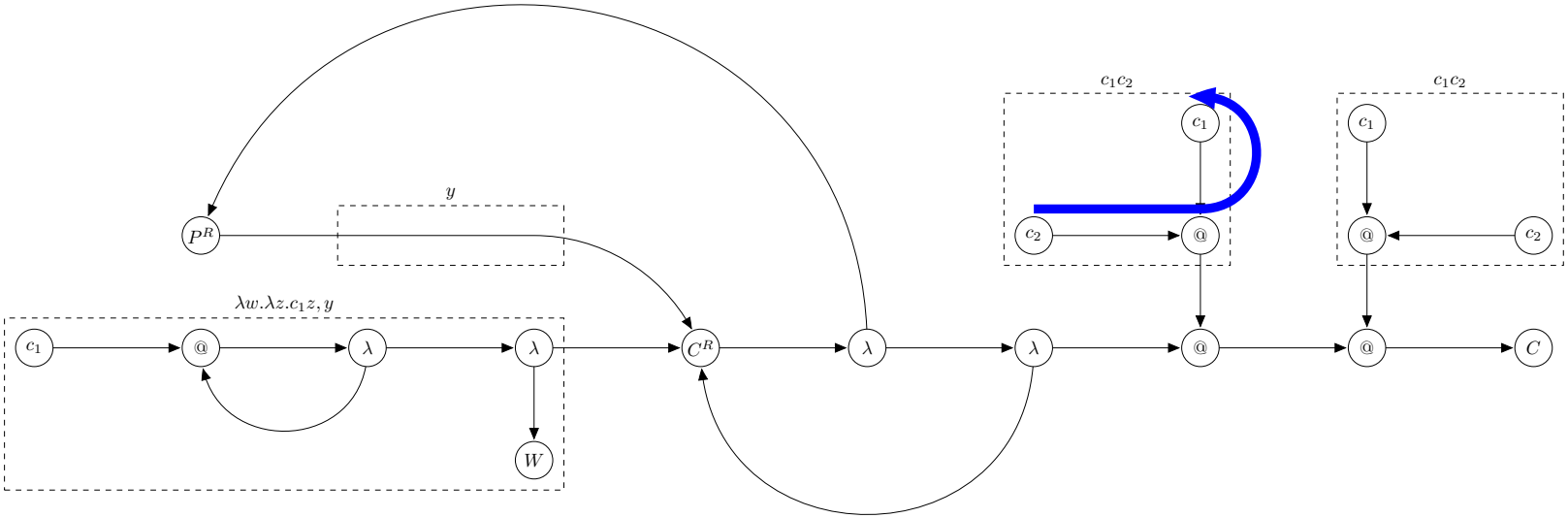

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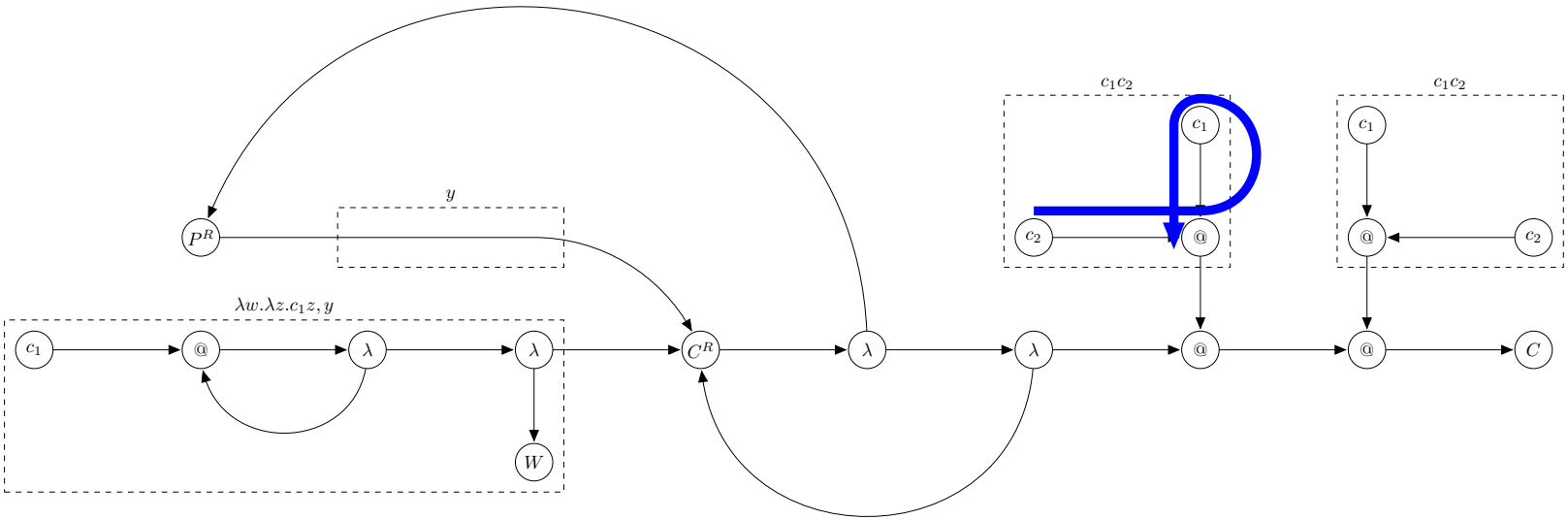

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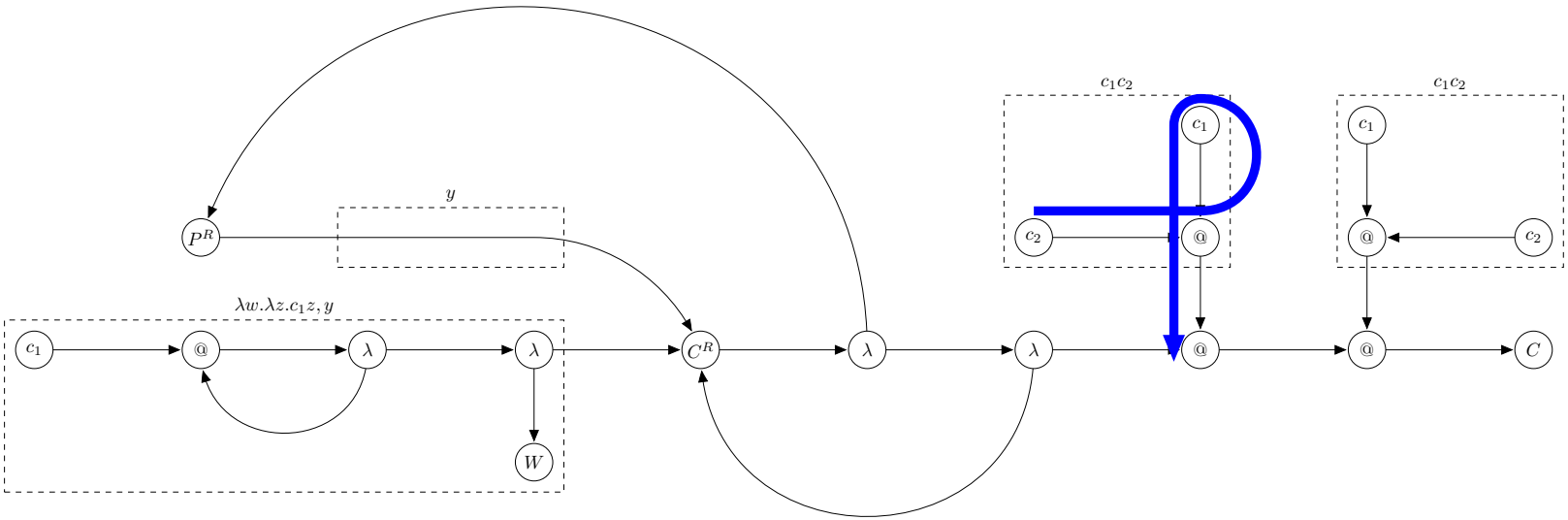

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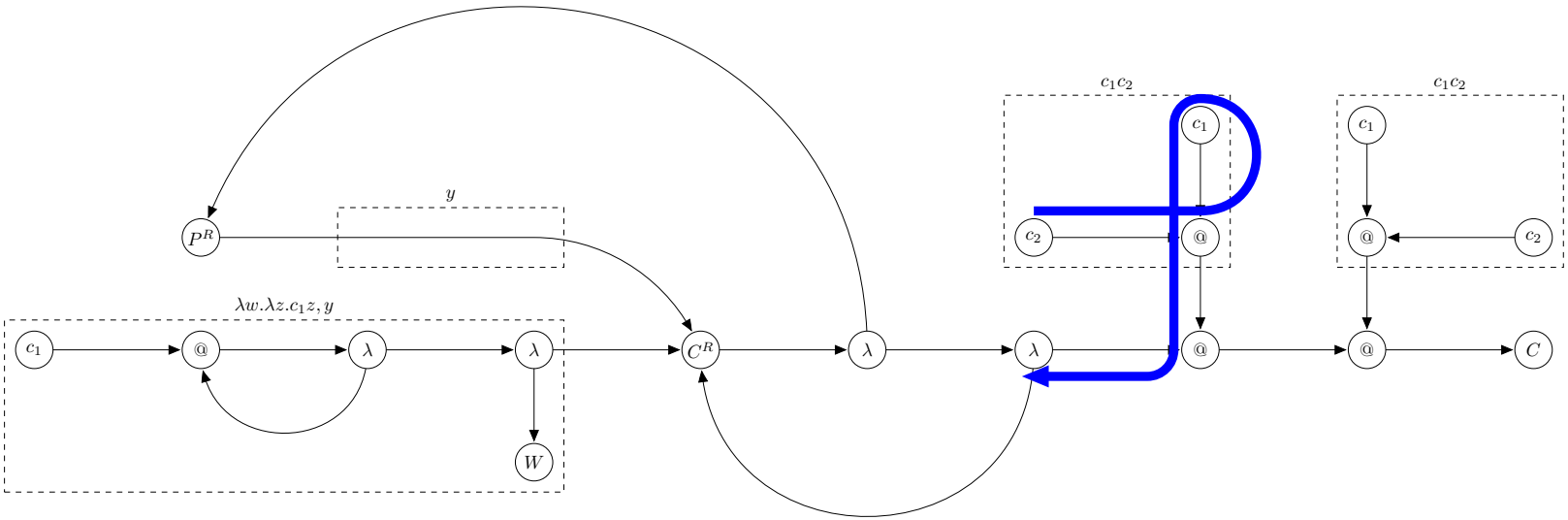

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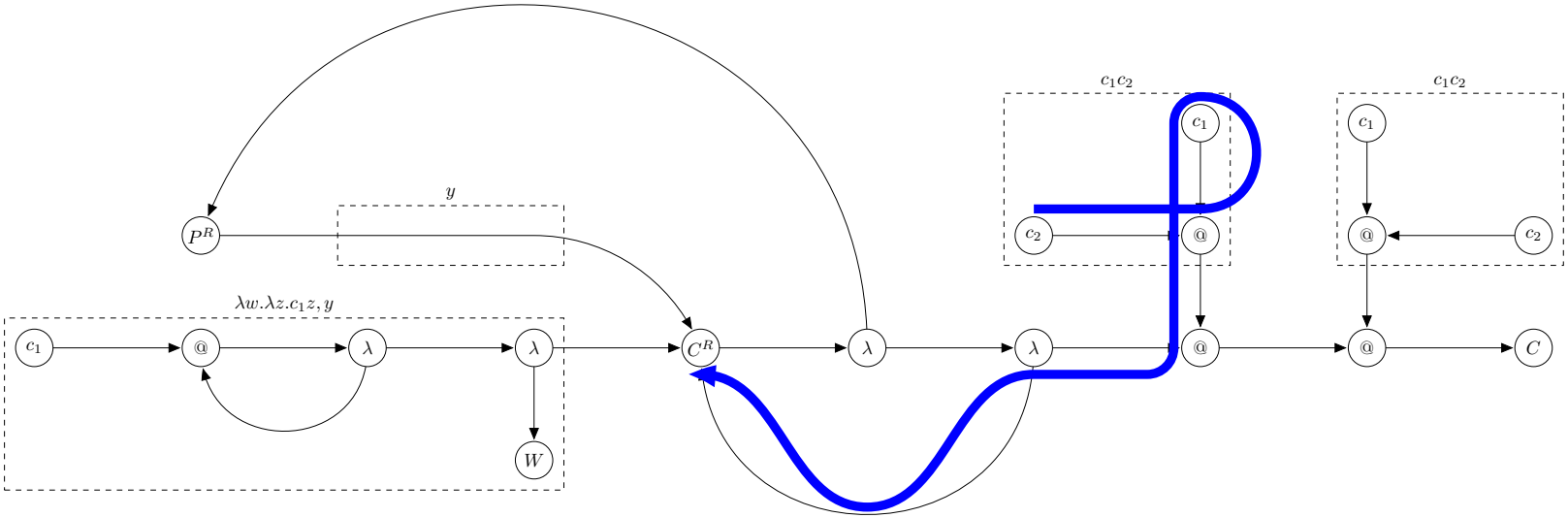

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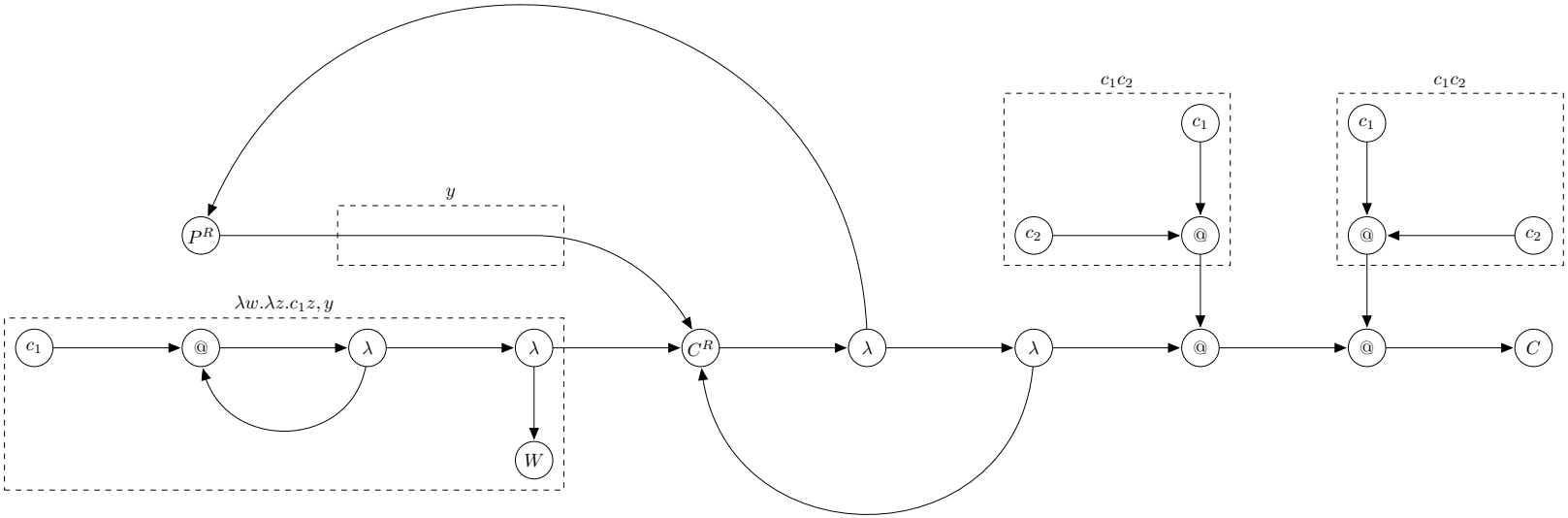

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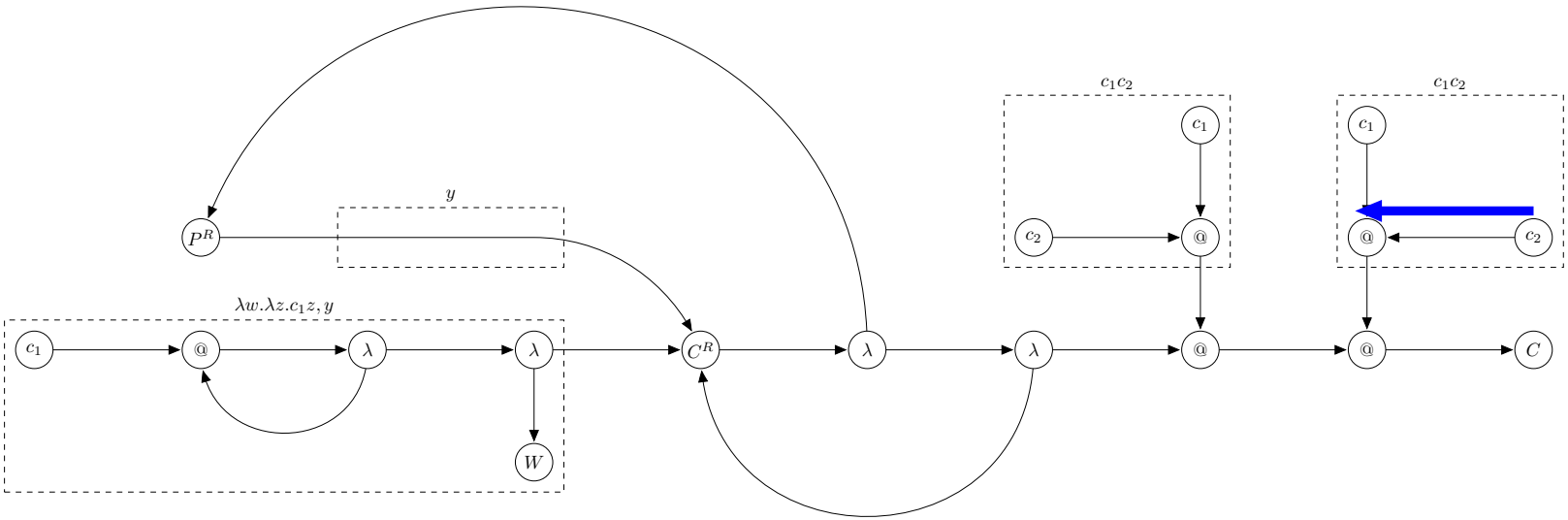

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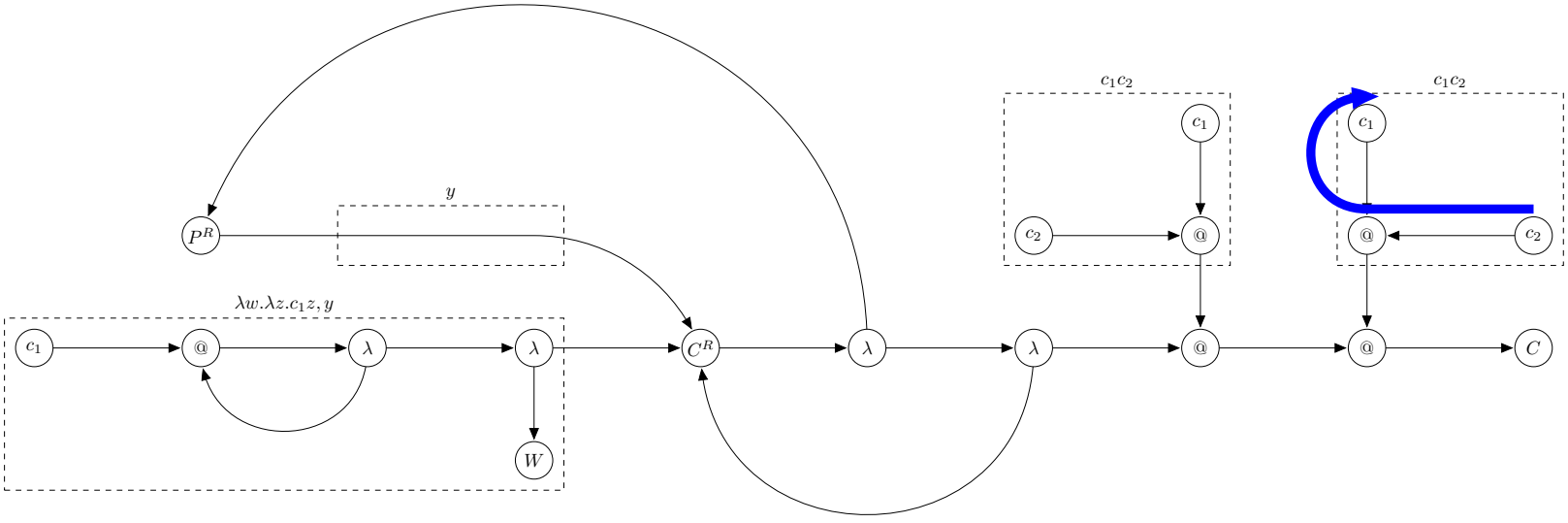

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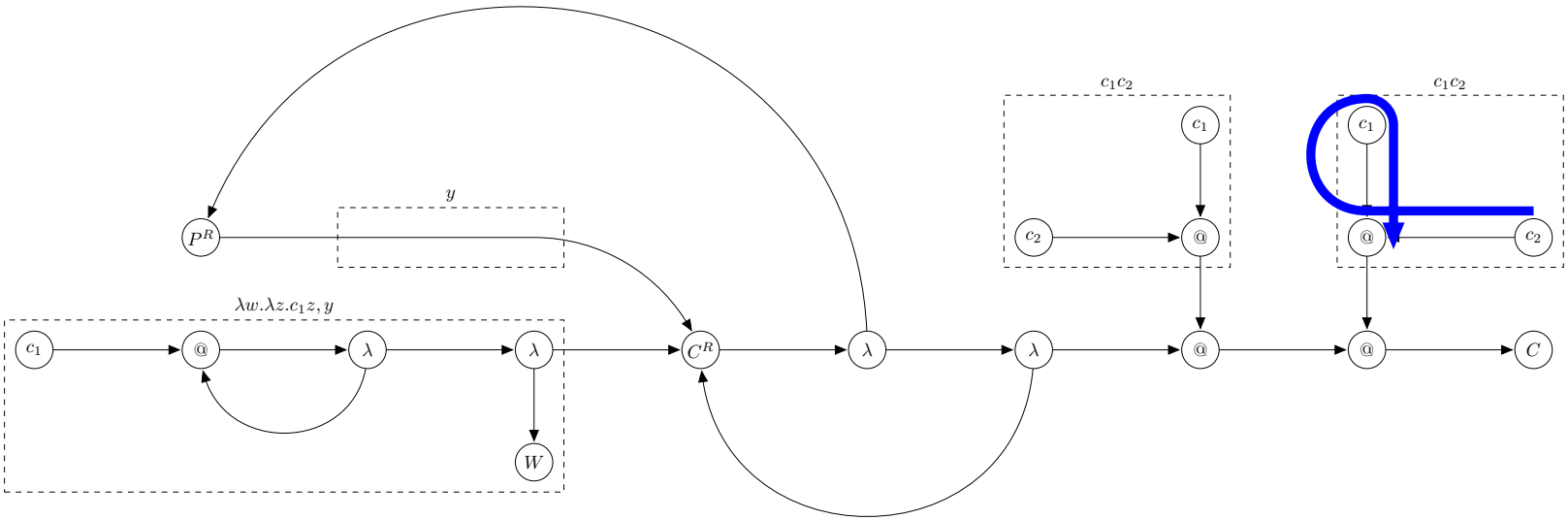

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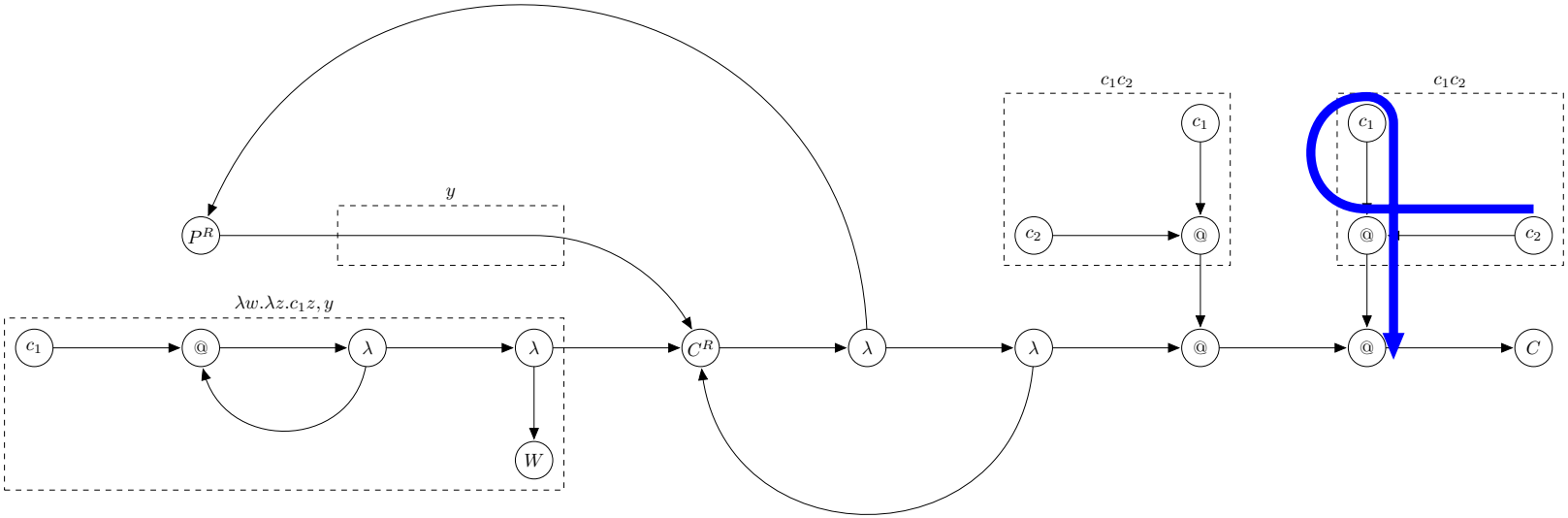

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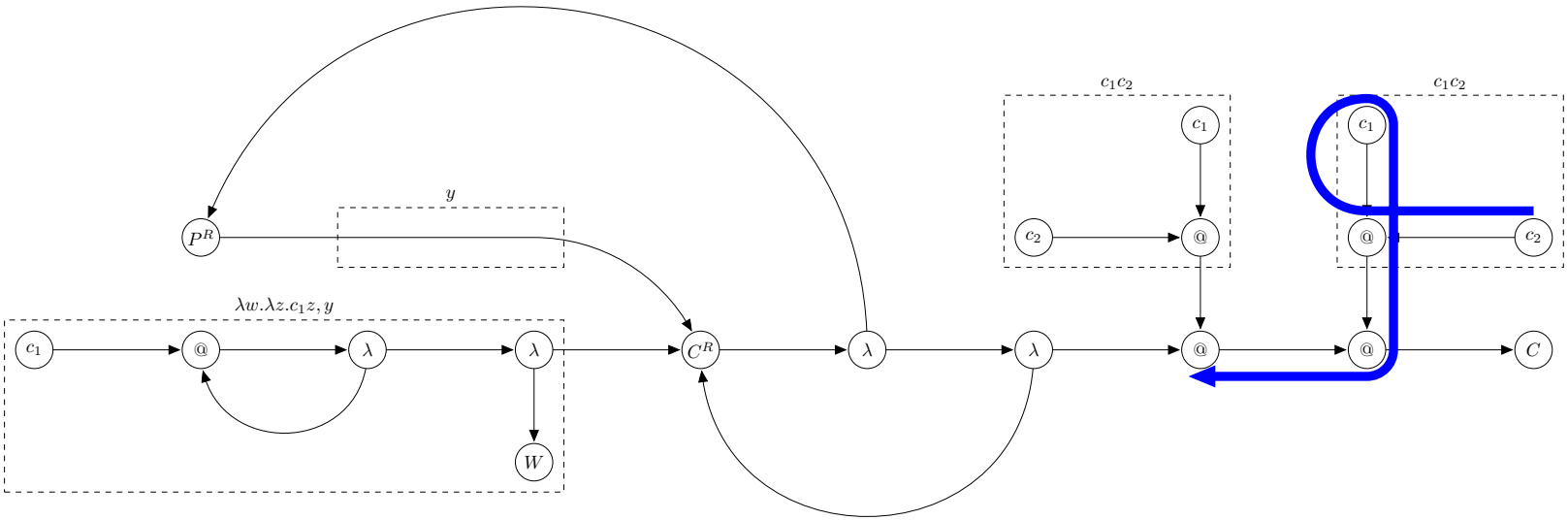

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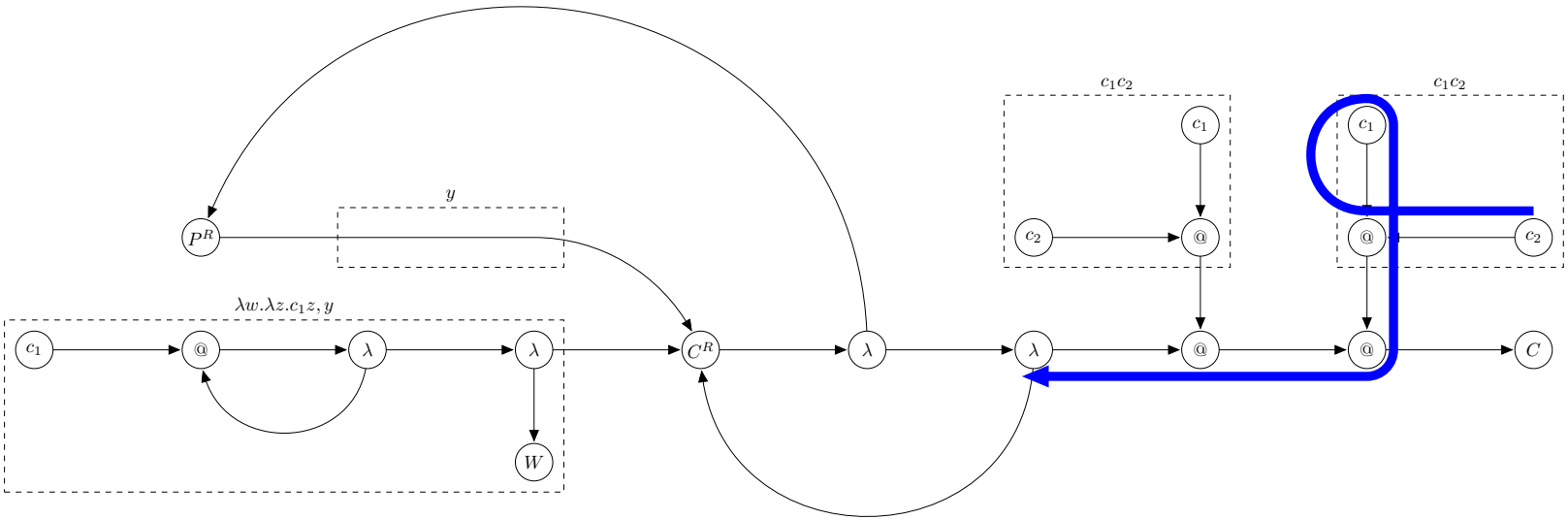

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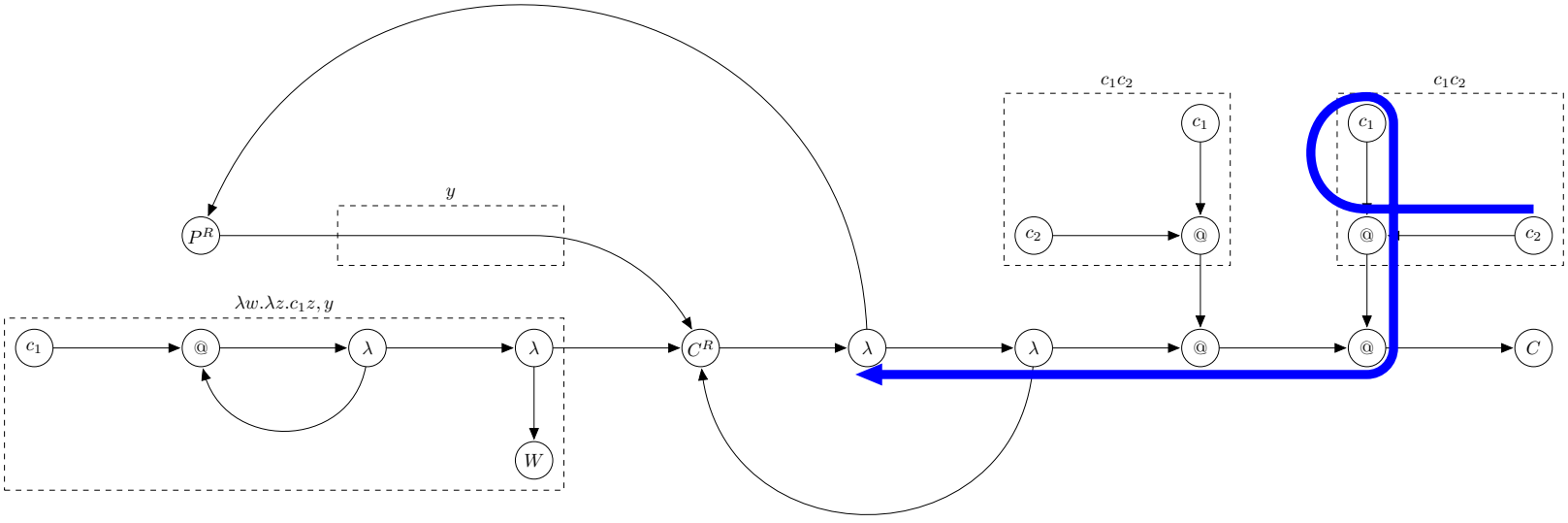

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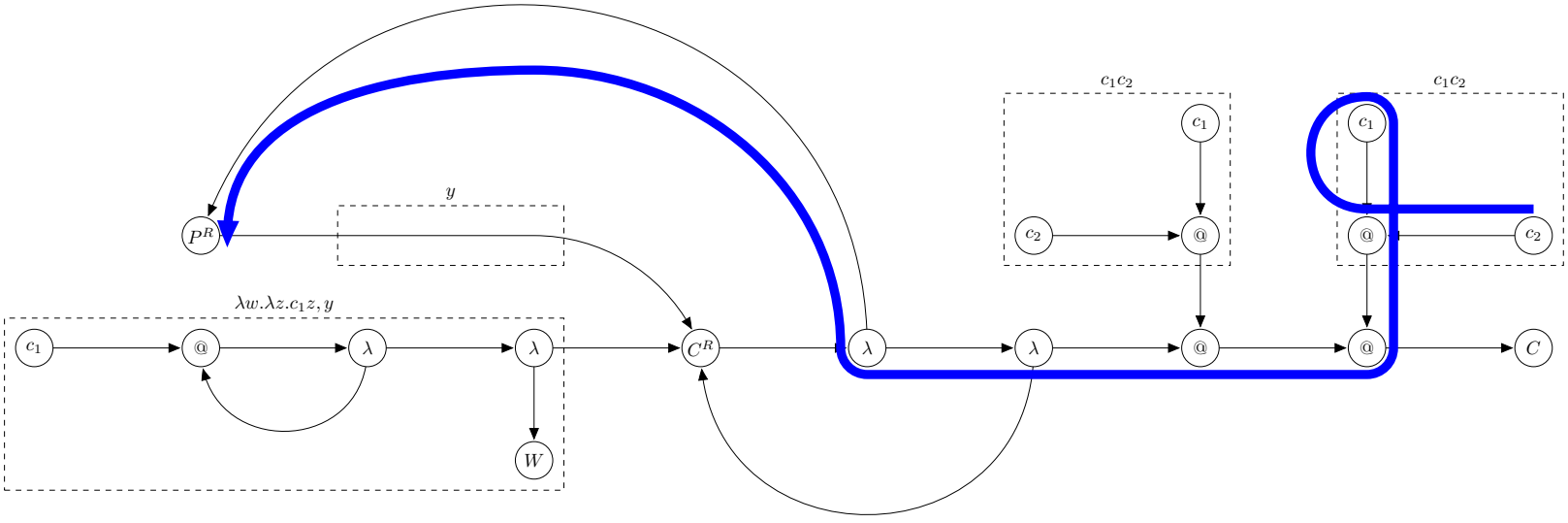

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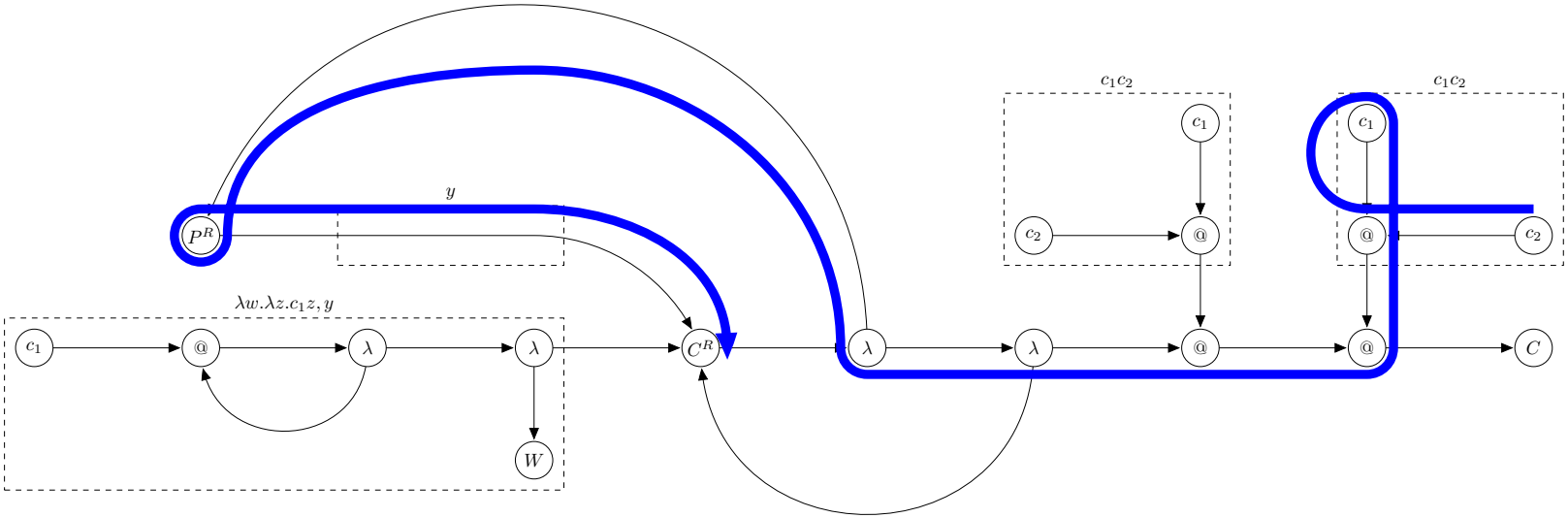

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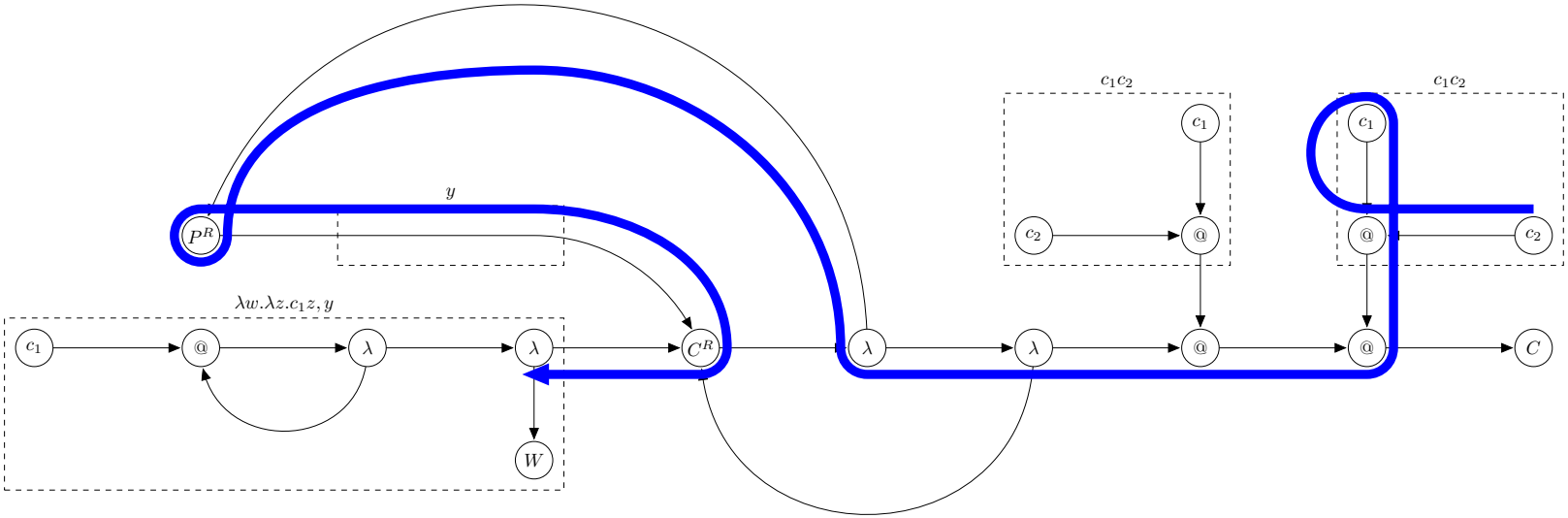

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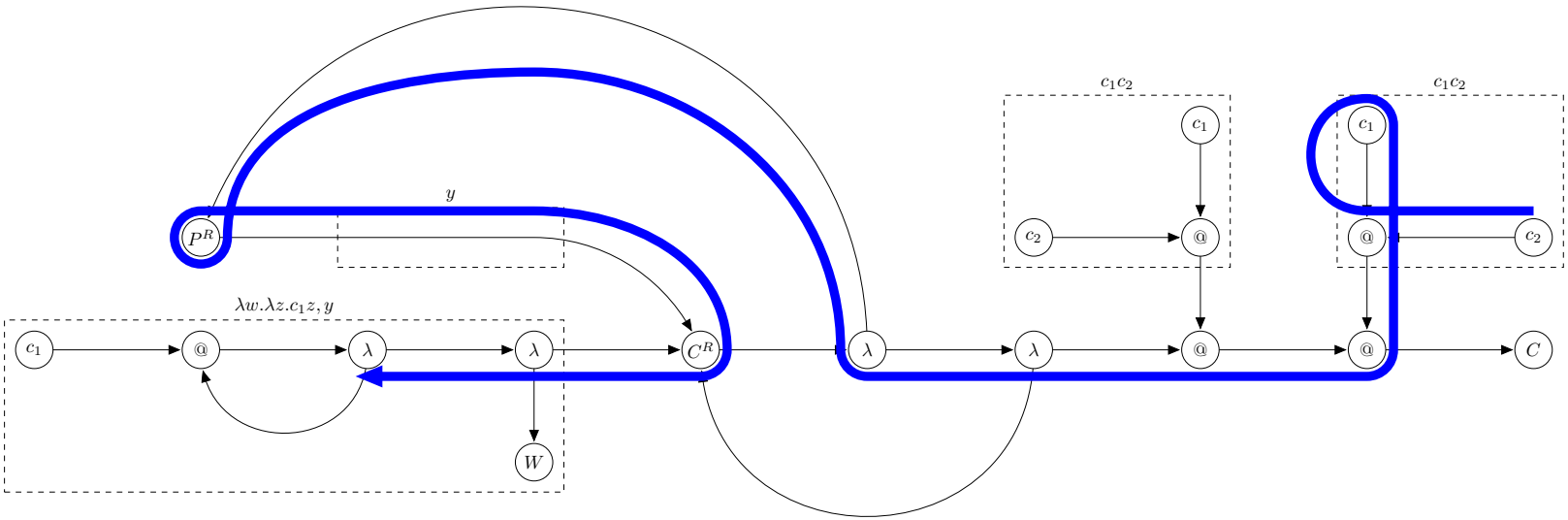

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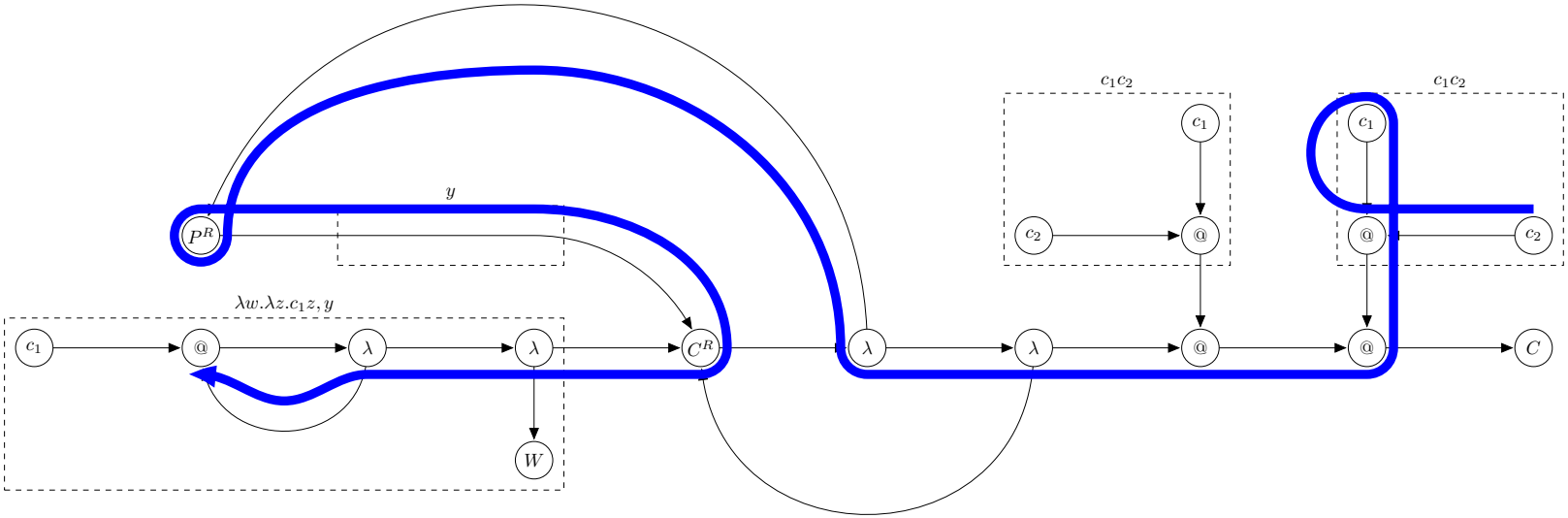

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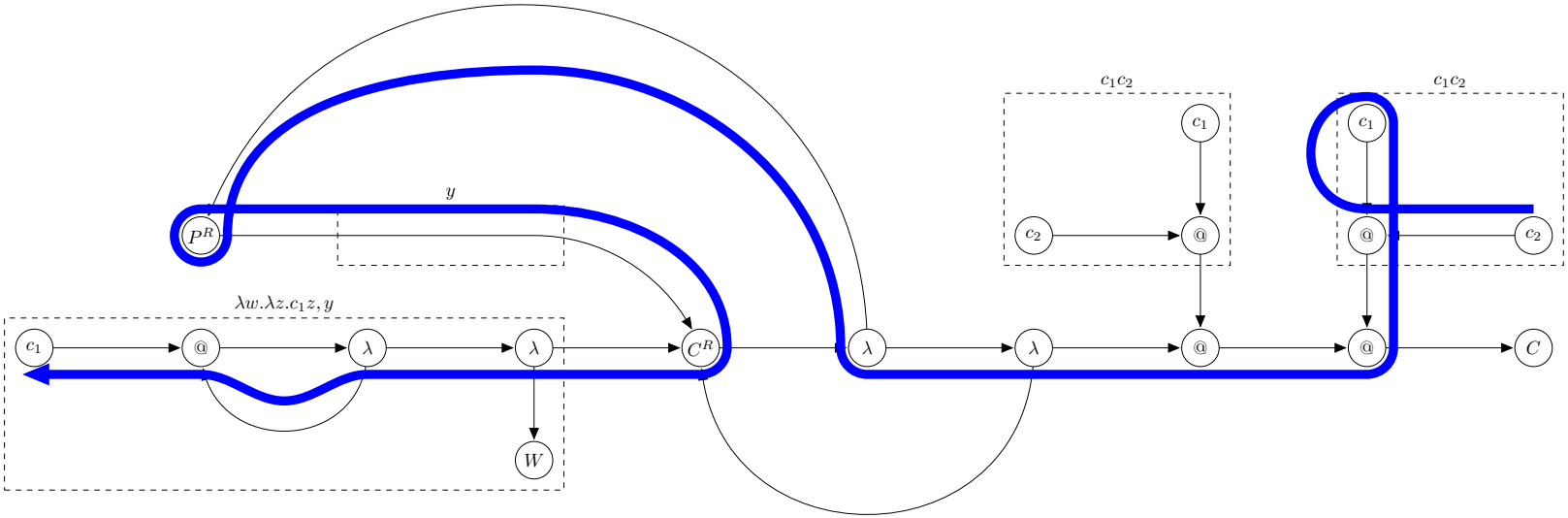

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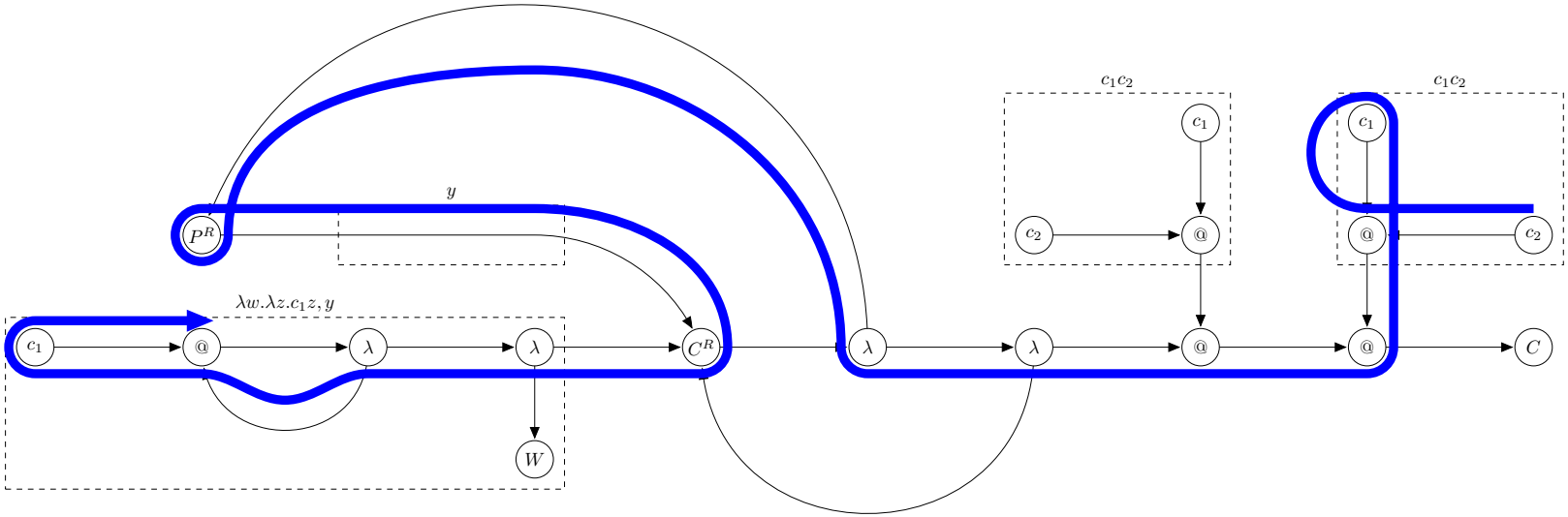

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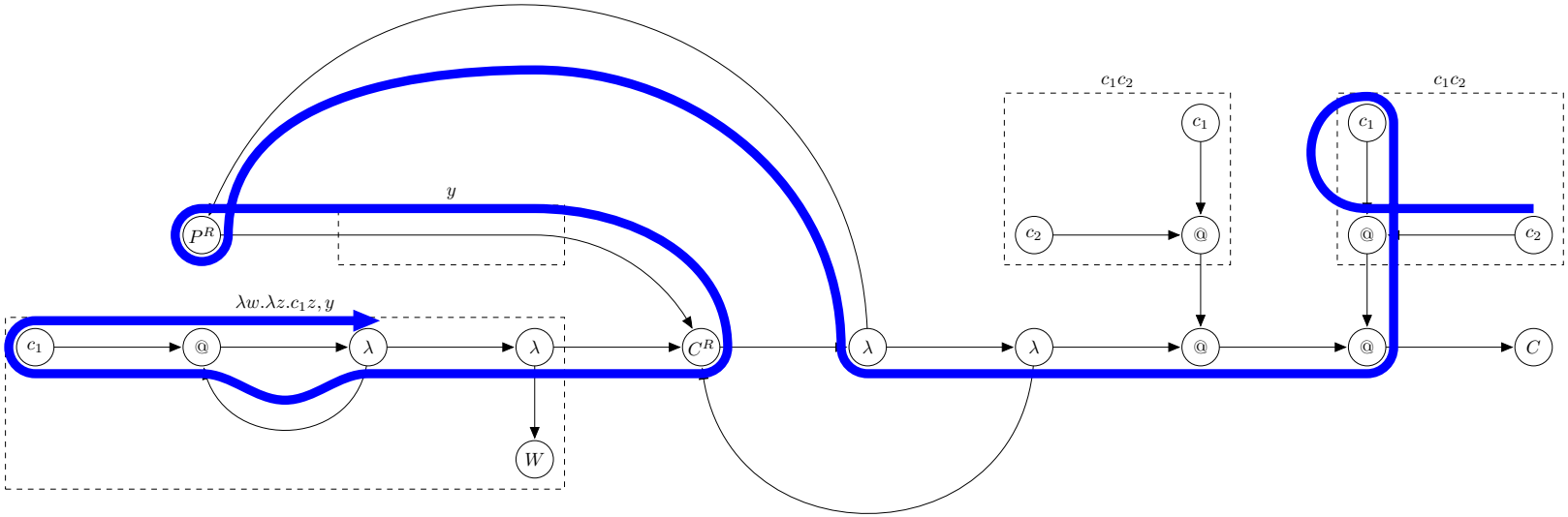

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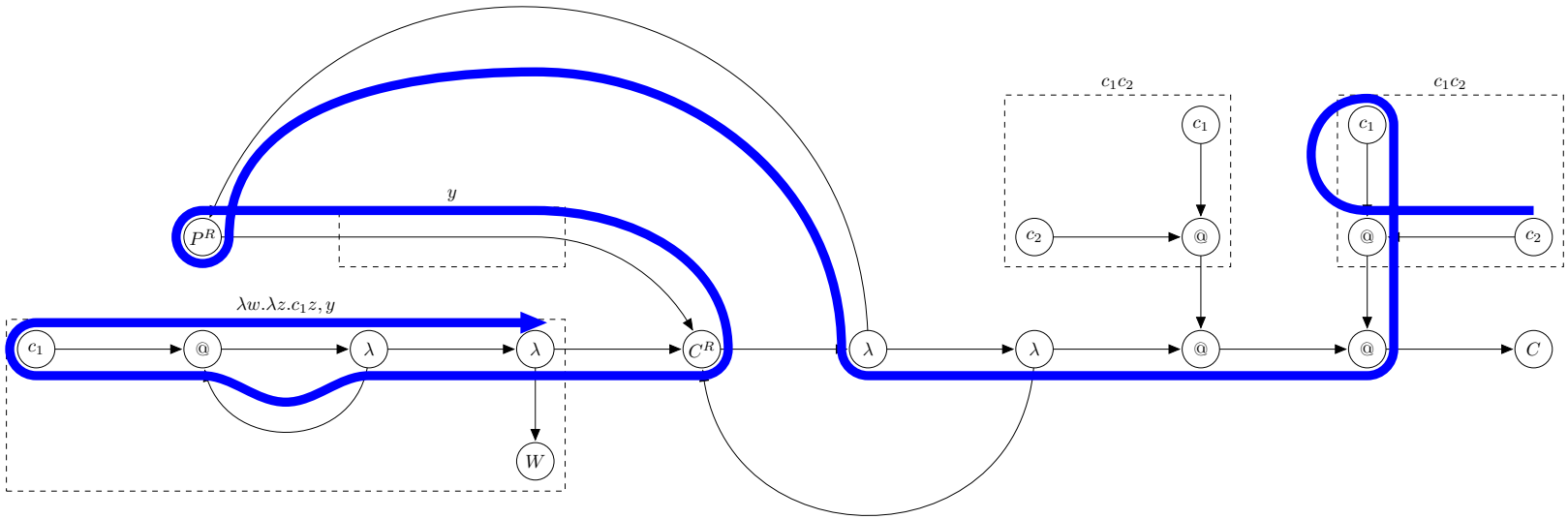

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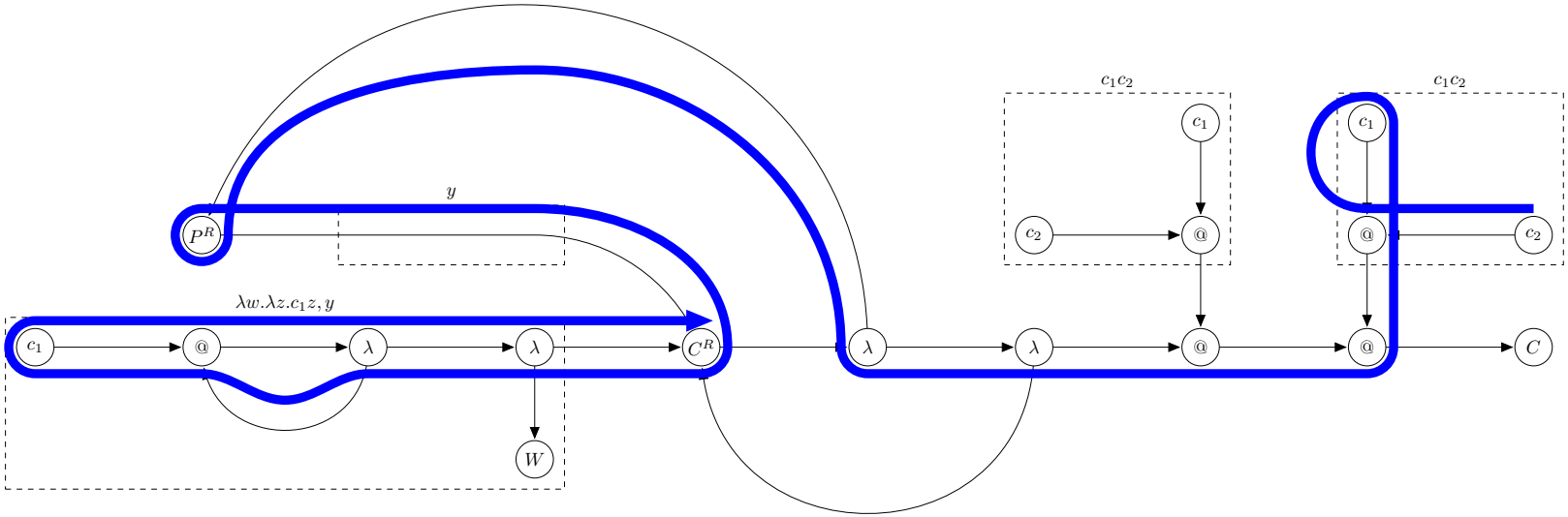

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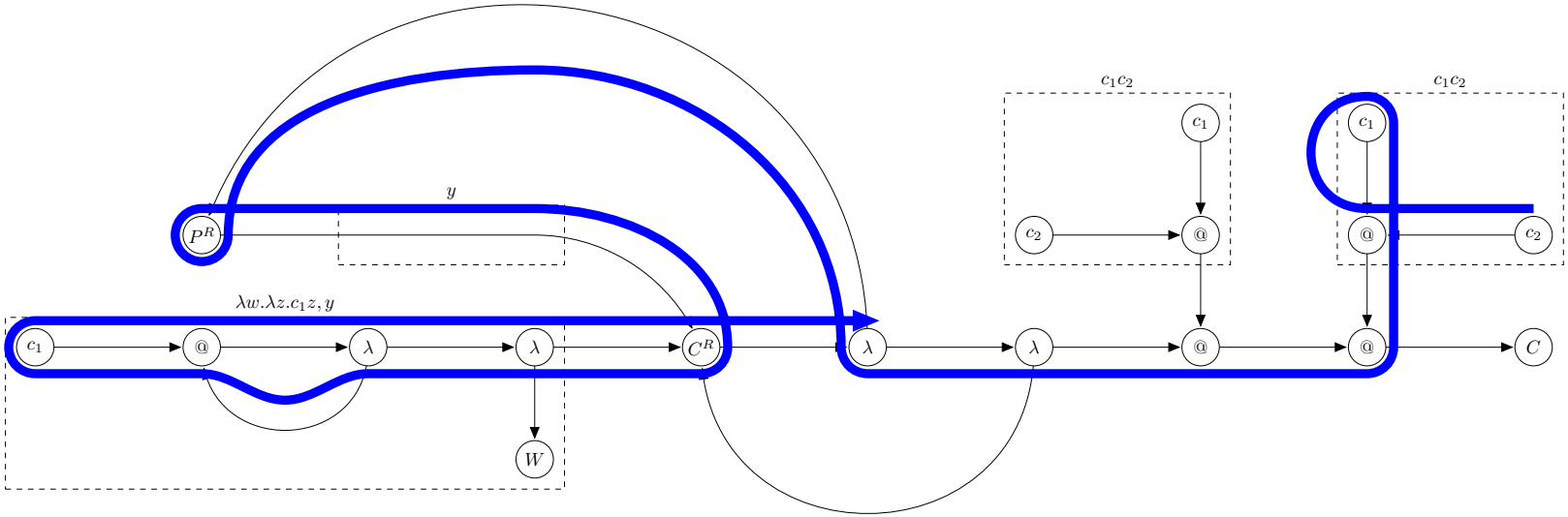

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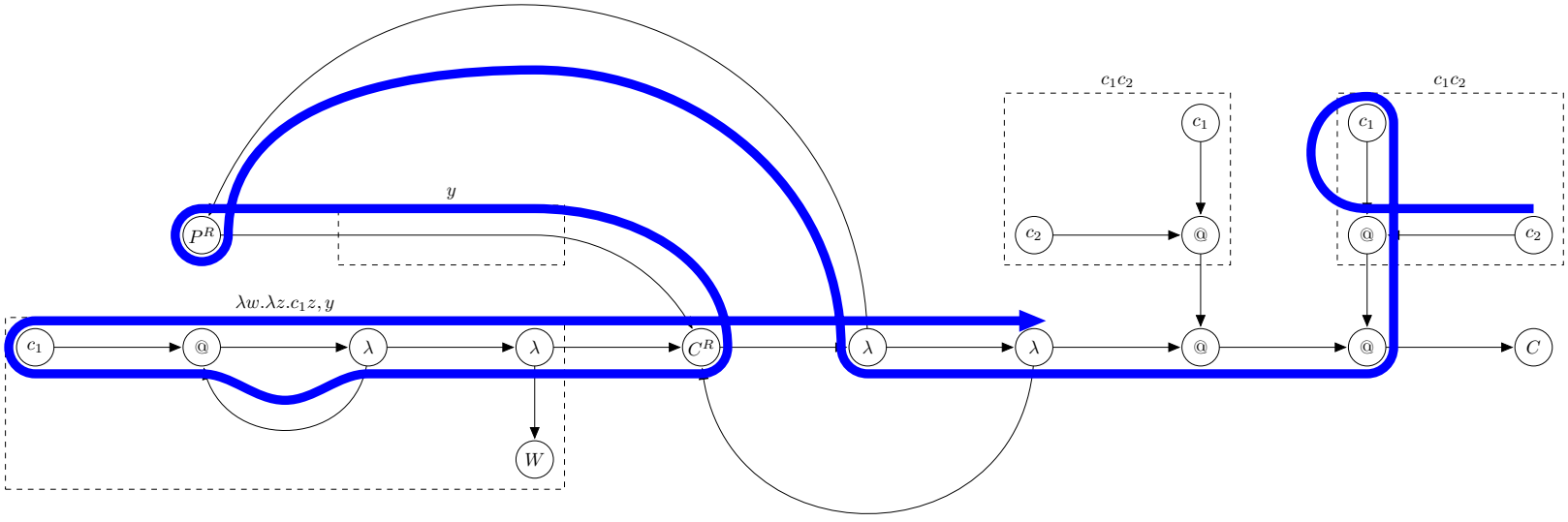

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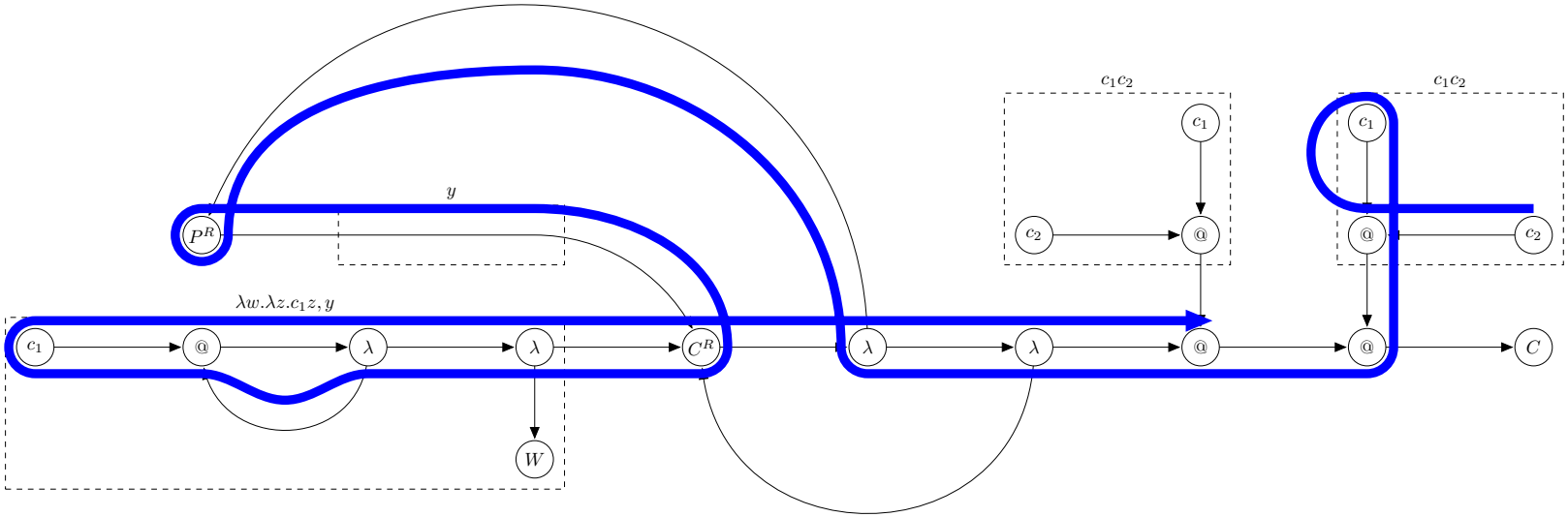

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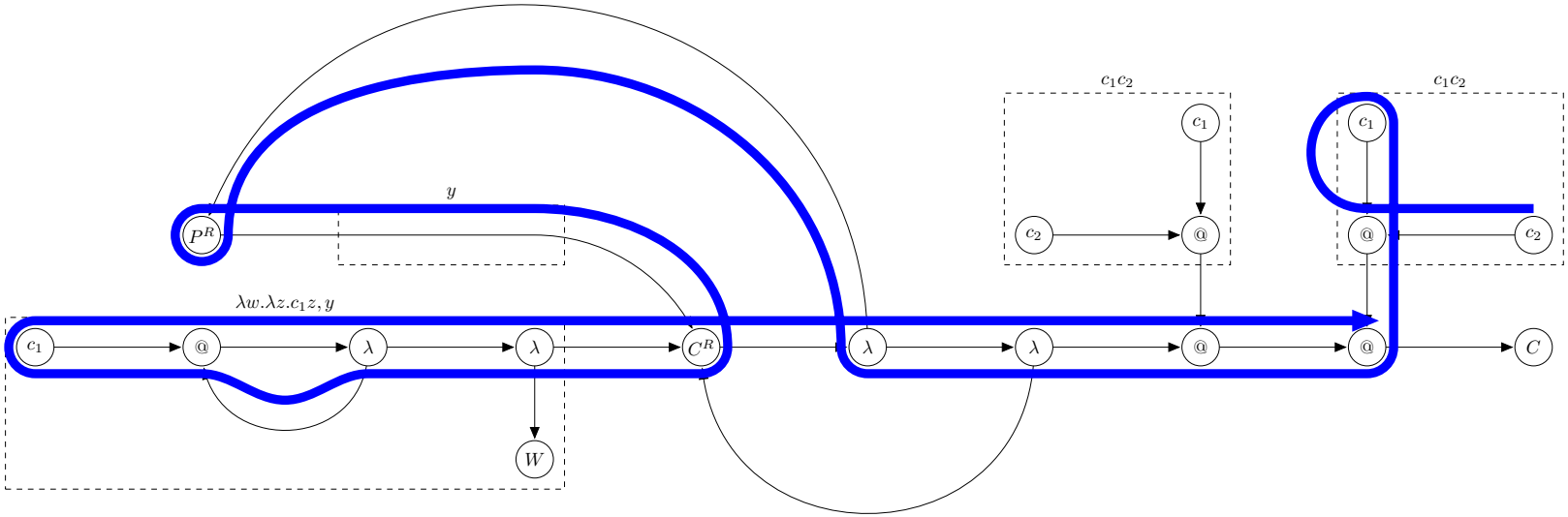

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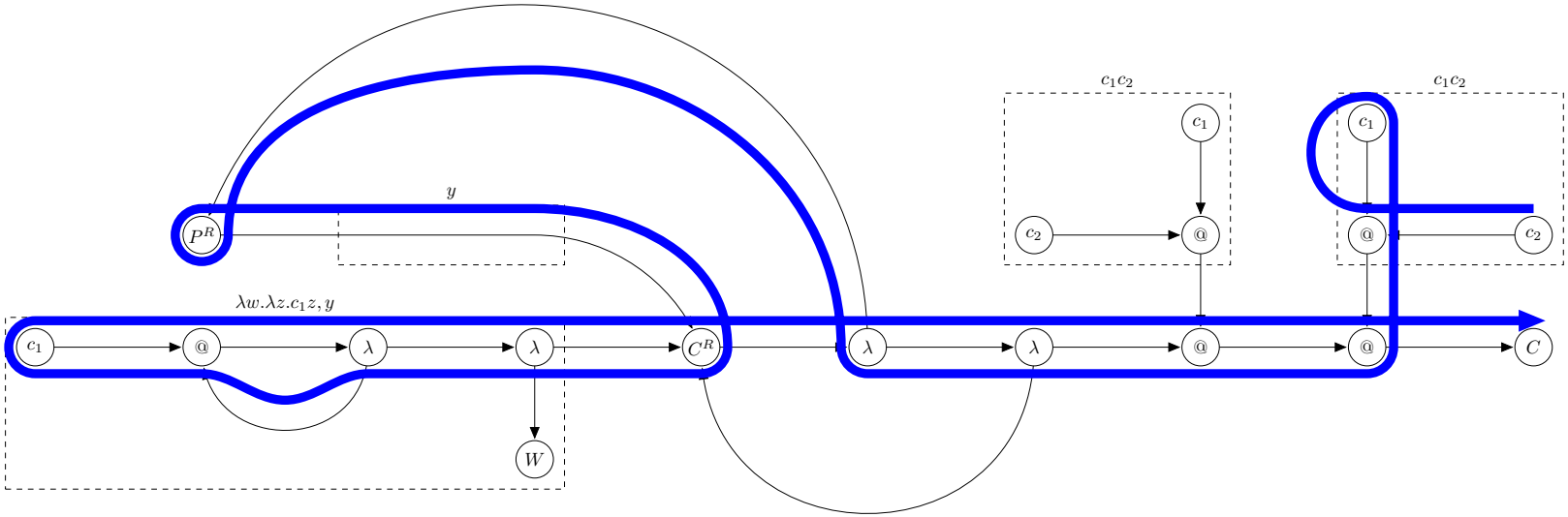

Algebraic Context Semantics is Enough

- First of all, we can give the following

Lemma (Adequacy)

If $\pi : \Gamma \vdash M : A$ and M contains a redex with argument t, then there is $T \in \mathcal{T}(G_\pi)$ with $\Phi(T) = t$.

- Moreover:

Lemma (Backward Preservation)

If $\pi : \Gamma \vdash M : A$ and $M \rightsquigarrow N$, there is $\xi : \Gamma \vdash N : A$ such that whenever $T \in \mathcal{T}(G_\xi)$ there is $U \in \mathcal{T}(G_\pi)$ with $\Phi(T) = \Phi(U)$.

- As a consequence:

Theorem

If $\pi : \Gamma \vdash M : A$, then there is $T \in \mathcal{T}(G_\pi)$ with $|\Phi(T)| = A(M)$.

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Keeping Everything Under Control

- First of all, there is a uniqueness property:

Lemma (Uniqueness)

For every context $(A[\cdot], C)$, there is at most one tree $T \in \mathcal{T}(G)$ whose root is labelled with $(A[\cdot], C)$.

As a consequence, any context can appear at most once in a given tree T.

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What Remains to be Done?

- We only need to analyze how restrictions to π reflect to $\mathcal{T}(G_\pi)$.
- In particular, we prove bounds on the size of elements of $\Phi(\mathcal{T}(G_\pi))$.
- Relatively easy, exploiting induction on trees.
- For every fragment, a few lemmas suffice.

Syntax

- Formulae:

$$A ::= \alpha \mid A \multimap A \mid A \otimes A \mid !A \mid \forall \alpha.A$$

- Rules:

$$\frac{}{A \vdash A}\ A \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ U \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ W \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ X$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ R_{\multimap} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ L_{\multimap}$$

$$\frac{A_1, \ldots, A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ P_! \qquad \frac{A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ D_! \qquad \frac{!!A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ N_!$$

$$\frac{\Gamma \vdash A \quad \alpha \notin FV(\Gamma)}{\Gamma \vdash \forall \alpha.A}\ R_{\forall} \qquad \frac{\Gamma, A\{B/\alpha\} \vdash C}{\Gamma, \forall \alpha.A \vdash C}\ L_{\forall}$$

Proof-nets

- We adopt a system of intuitionistic proof-nets.
- For example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,A \quad \dfrac{}{\alpha \vdash \alpha}\,A}{\alpha, \alpha \multimap \alpha \vdash \alpha}\,L_! \quad \dfrac{}{\alpha \vdash \alpha}\,A}{\alpha, \alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,L_!}{!\alpha, \alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,D_!}{!\alpha, !\alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,D_!}{!\alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,X}{\alpha \multimap \alpha \multimap \alpha \vdash !\alpha \multimap \alpha}\,R_! \qquad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,A \quad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,A}{!\alpha \vdash !\alpha}\,P_!}{!\alpha \multimap \alpha, !\alpha \vdash \alpha}\,L_!}{\alpha \multimap \alpha \multimap \alpha, !\alpha \vdash \alpha}\,U$$

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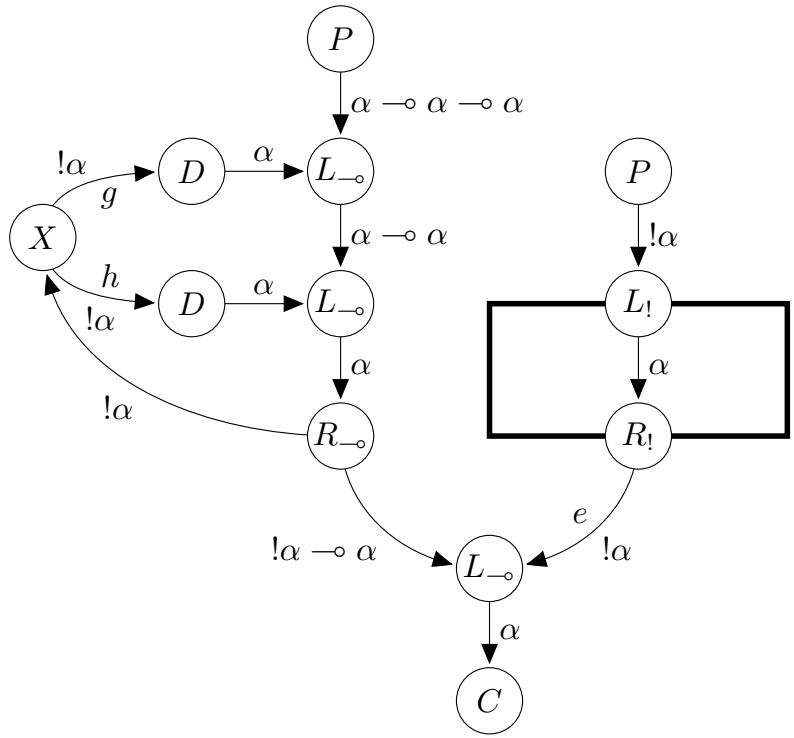


Copying - The Root of Complexity?

- Linear logic without exponentials and additives cannot express anything beyond polynomial time.
- Suppose we know the total number of times boxes in a proof-net G can (possibly) be duplicated. Call it W_G.
- Notice W_G can even be infinite.
- How W_G relates to the complexity of normalizing G?
- Context semantics helps in giving strong answers to this question.

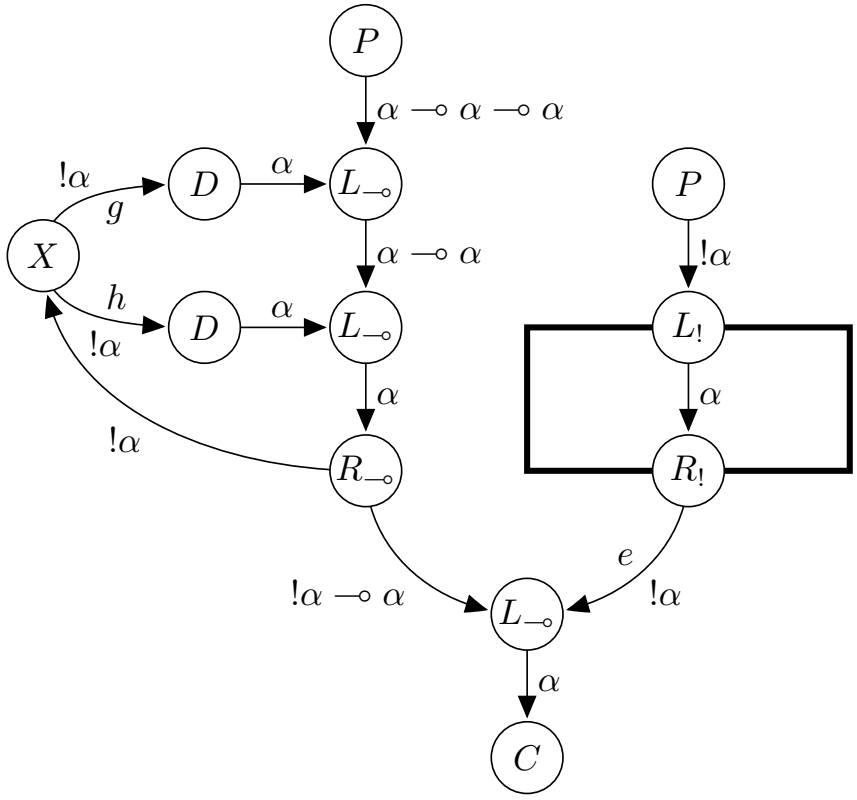
Context Semantics in Five Minutes

- Every proof-net G induces rewriting rules on

$$E_G \times C_G$$

  where E_G is the set of G edges, and C_G is a set of *contexts* for G.
- If $(e_1, C_1) \to (e_2, C_2) \to \ldots \to (e_n, C_n)$, then $e_1, \ldots, e_n$ is a *persistent path*.
- There are certain contexts called *exponential trees*.
- If e is a box main premise, then every exponential tree t with $(e, t) \to \ldots \to (g, u)$ (where u is somehow minimal) corresponds to a different "copy" of the box.
- W_G as the sum over all boxes of the number of exponential trees satisfying the above condition.

An Example



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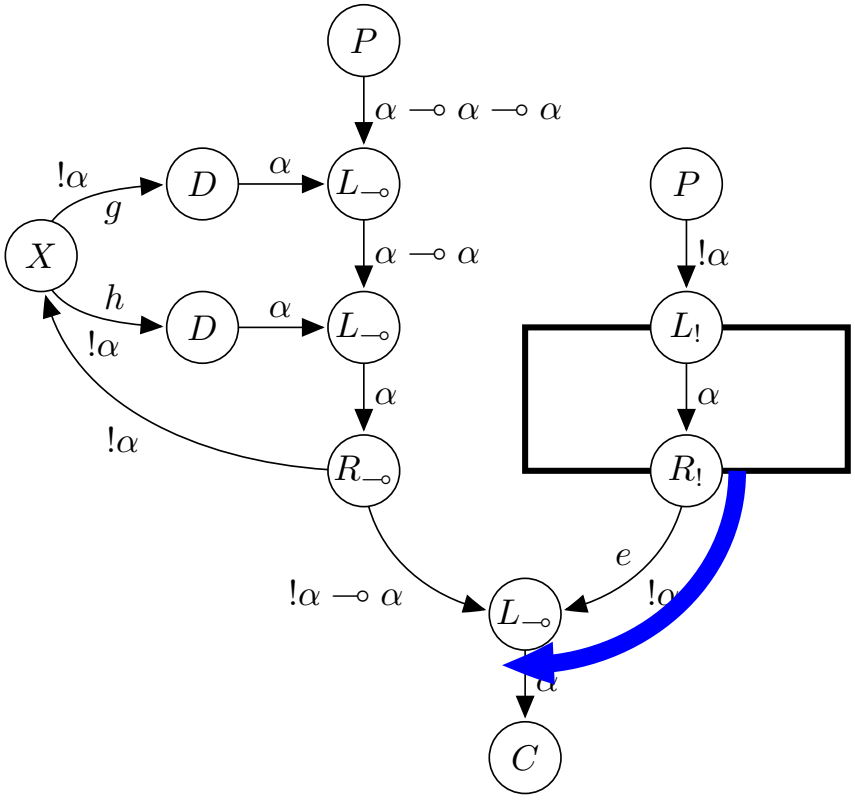

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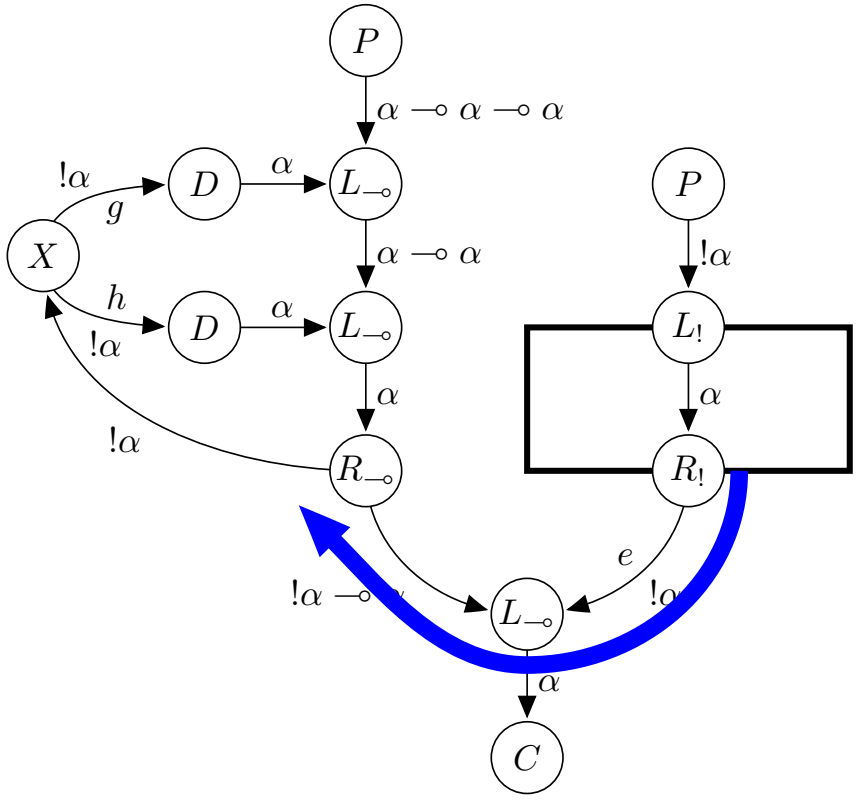

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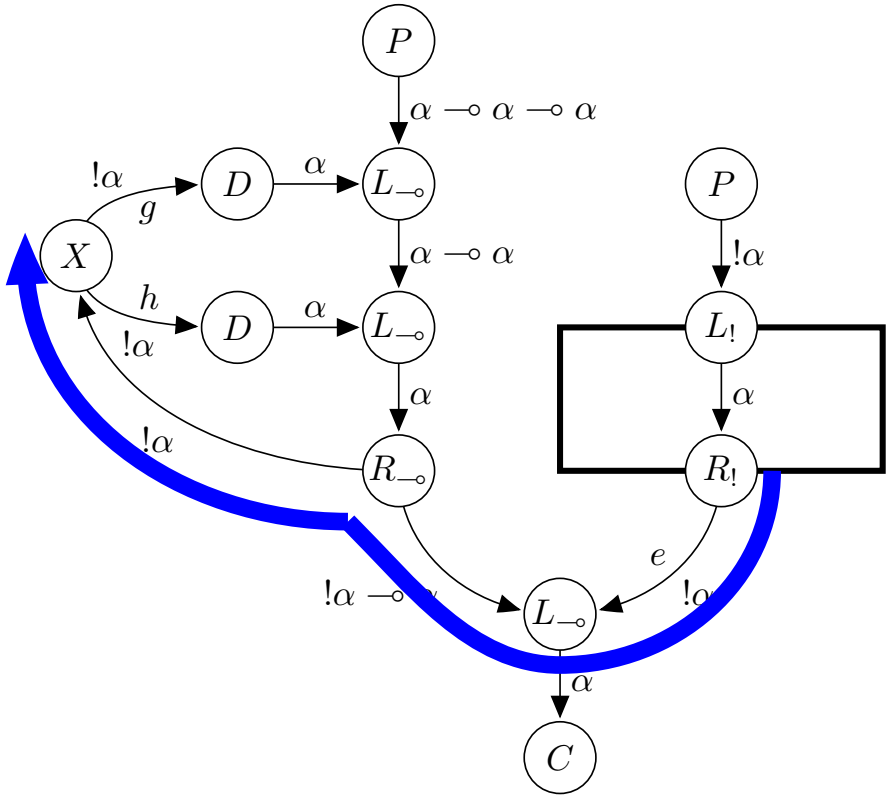
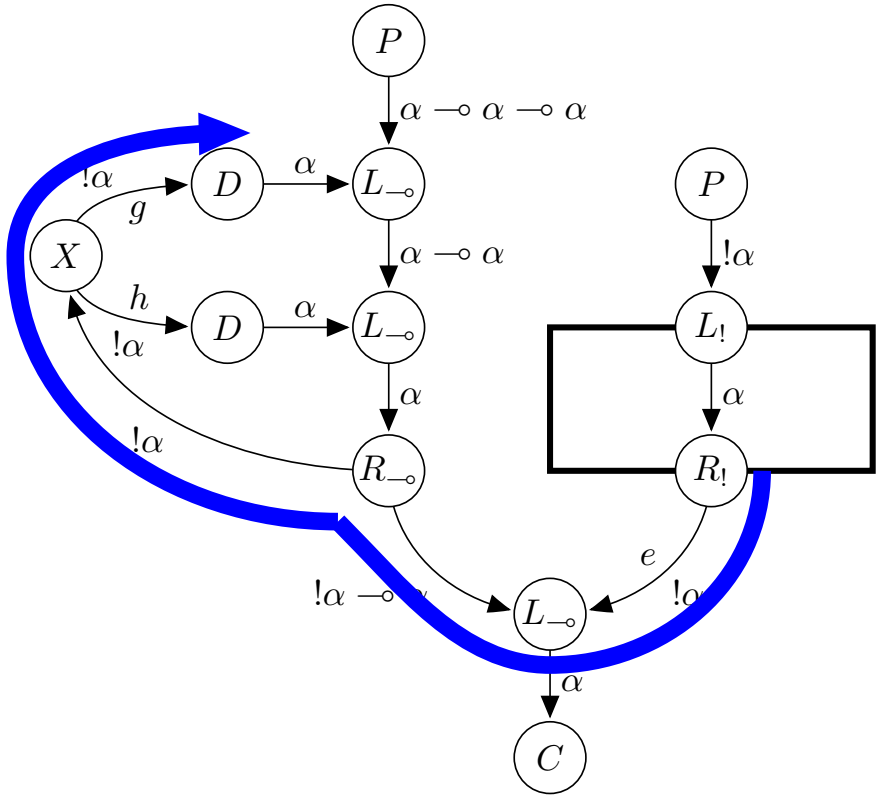


An Example



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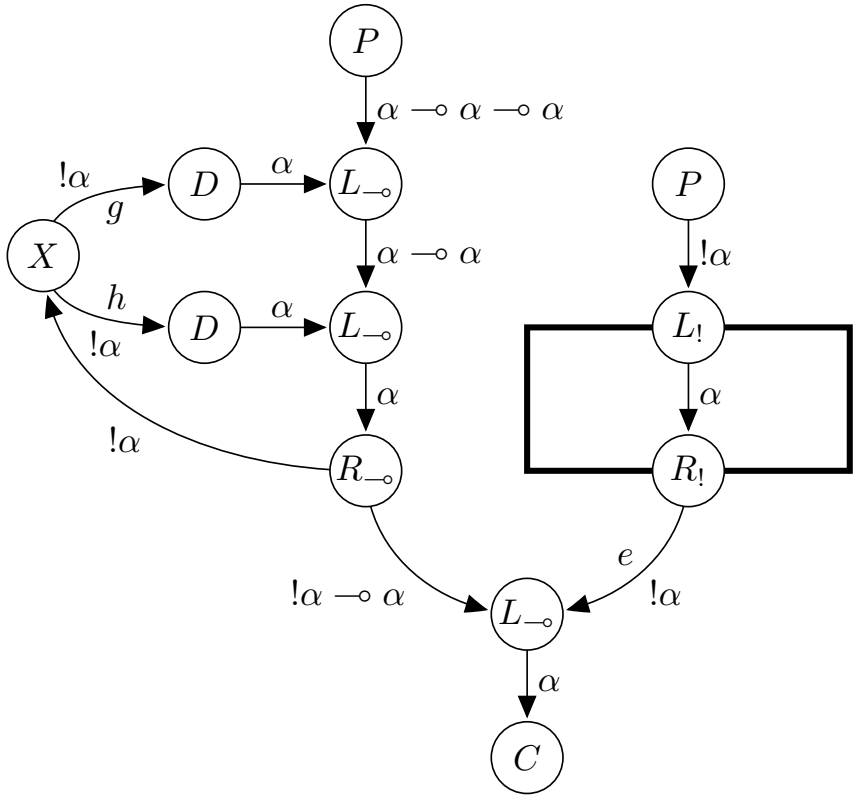

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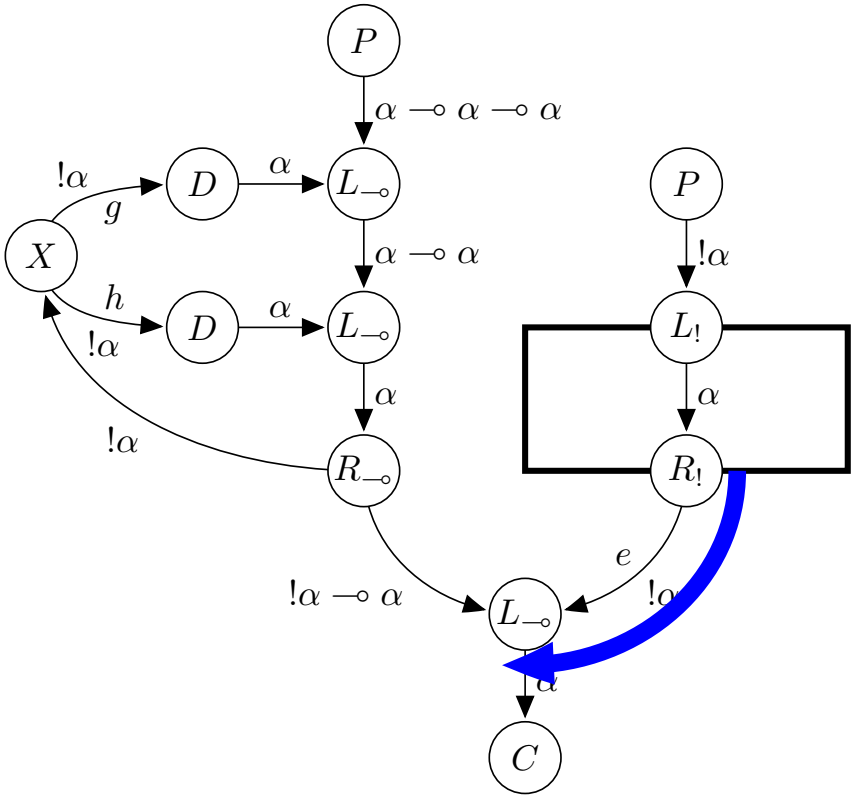

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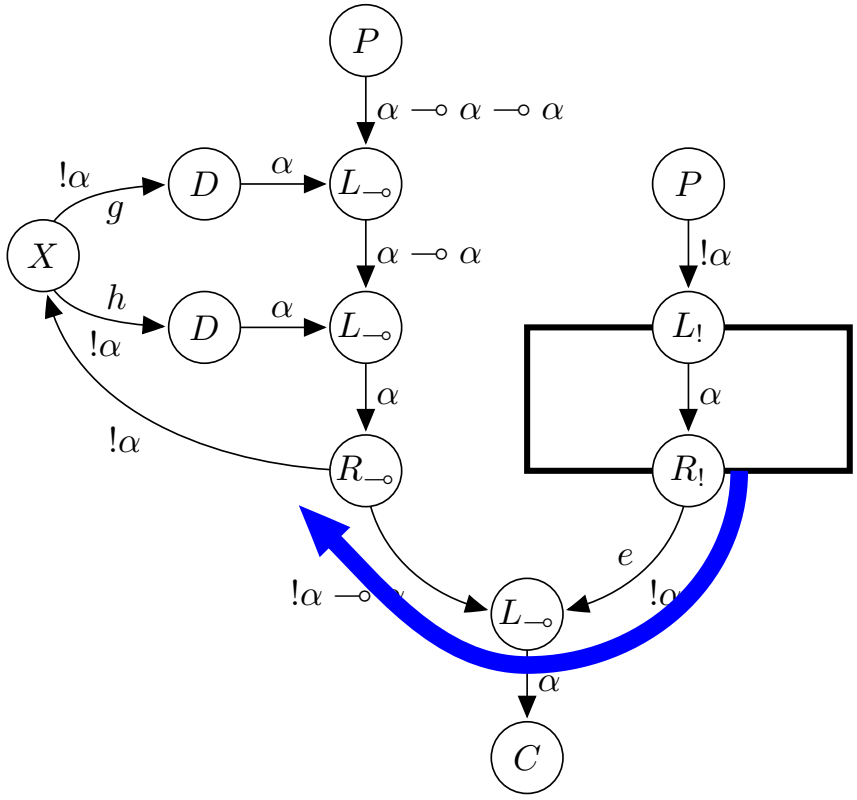

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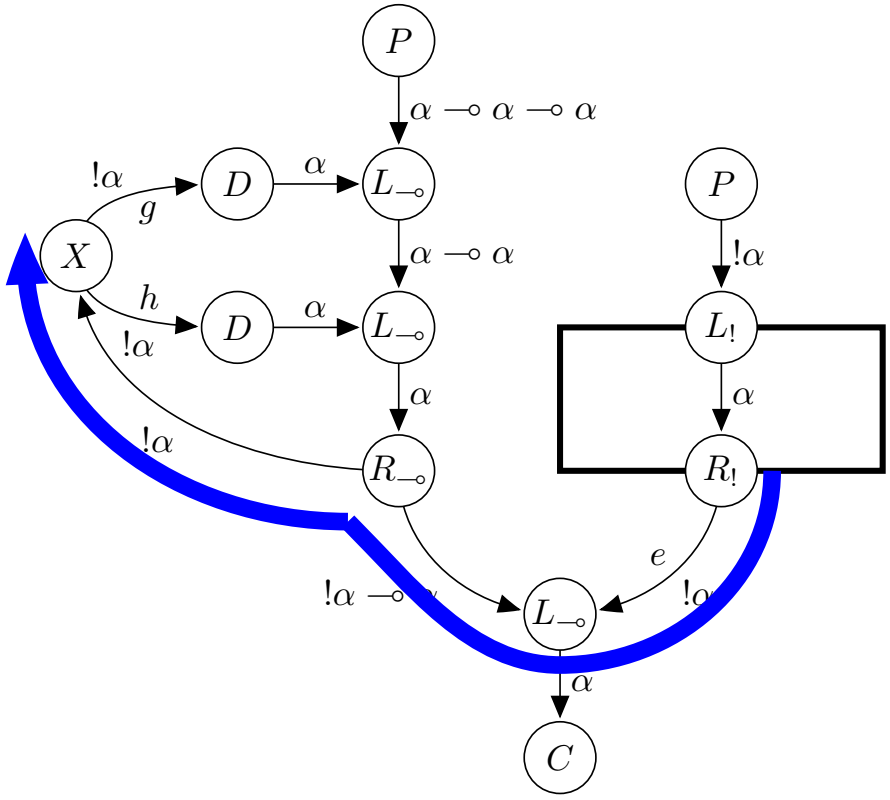
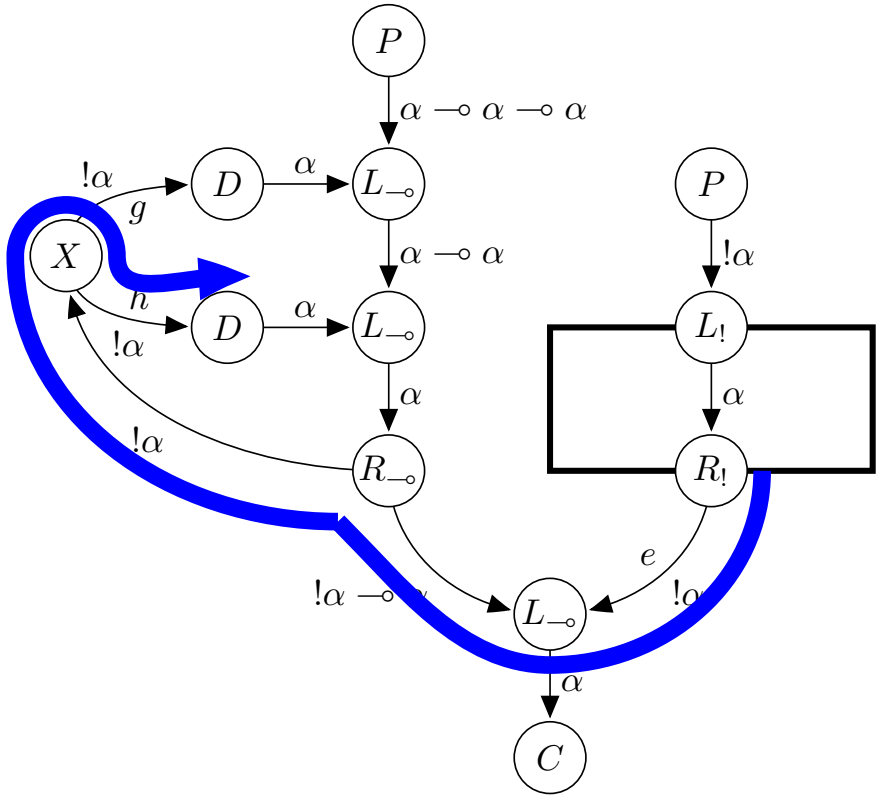


An Example



One Direction

- W_G is always positive, but is not guaranteed to strictly decrease at any normalization step.
- Moreover, it is not clear whether the size of $|G|$ is related to W_G or not.
- However, we can define from W_G and $|G|$ another quantity T_G which *both* decreases at every normalizatop step and majorizes $|G|$.
- As a consequence, we get:

Theorem

There is a polynomial $p : \mathbb{N}^2 \to \mathbb{N}$ such that for every proof-net G, G normalizes in at most $p(W_G, |G|)$ steps and the size of any reduct H of G is at most $p(W_G, |G|)$.

The Other Direction

- Is W_G a gross overestimate on the time needed to normalize G?
- Consider the *level-by-level* strategy:
  - For every $n \in \mathbb{N}$, a cut at level $n+1$ is fired only if there are not cuts at levels from 1 to n.
  - For every $n \in \mathbb{N}$, a !-cut at level n is fired only if every cut at level n is either a W-cut or a !-cut
  - A W-cut is fired only if every cut in the proof-net is a W-cut.
- W_G decreases by **at most one** at every step.
- As a consequence, we get:

Theorem

Let G be a proof-net. If $W_G = \omega$, then G diverges, otherwise there is H with $G \rightarrow^{W_G} H$.

Contributions

- We have proved **several charaterization results** for the class of representable functions in fragments of higher-order recursion.
- Most work is **factored over** the various fragments and done just once.
- Copying has been proved to be **the real source** of complexity in the realm of linear logic.

Contributions

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Future Developments

- Other fragments of higher-order recursion (e.g. Non-size increasing polytime computation) seem to be analyzable in our framework.
- An interesting development consists in applying the methodology to Mackie's interaction nets for PCF.
- (Re)prove soundness theorems for linear logic fragments by way of context semantics.
- Is context semantics related to resource polynomials and weights in bounded linear logic?

Questions?