A jump from parallel to sequential proofs

Paolo Di Giamberardino

Dipartimento di Filosofia, Università Roma Tre

Institute de Mathématiques de Luminy

Joint work with Claudia Faggian

Introduction

In this talk we introduce *proof nets with jumps*, as a way to characterize proof nets with different degrees of sequentiality; we give a proof of the sequentialisation theorem for them, which turns out to be very simple (it doesn't use empires et *similia*), and we show how all this can be applied to the proof nets of MLL.

Information about sequentiality in a proof net

In a proof net we find two kind of sequentiality:

- Sequentiality given by the subformula trees;

- Sequentiality given by the axiom links.

-Question: What does it happen if we increase the sequentiality informations in a proof net?

-Answer: When we reach the top, we get a fully sequentialized proof.

Jumps

Jumps were introduced by Girard as a part of the correction criterion for MALL proof nets; a jump is a not typed arc starting from a negative link a to another link b, which expresses a *dependency* relation: a precedes b in the sequentialisation. This idea has been recently reprised by Faggian and Maurel in the abstract context of L-nets; due to the introduction of jumps, L-nets enjoys nice properties (for instance absence of boxes, cut elimination not "lazy", etc); one can think to transport this good properties to the "typed setting" (i.e. proof nets).

$MLL{\uparrow\downarrow}_{synt}$

$$\begin{array}{lcl} N & ::= & X^{\perp} \mid N \parr N \mid \uparrow P \\ P & ::= & X \mid P \otimes P \mid \downarrow N \end{array}$$

$$\frac{}{\vdash X,\ X^{\perp}}(Ax)$$

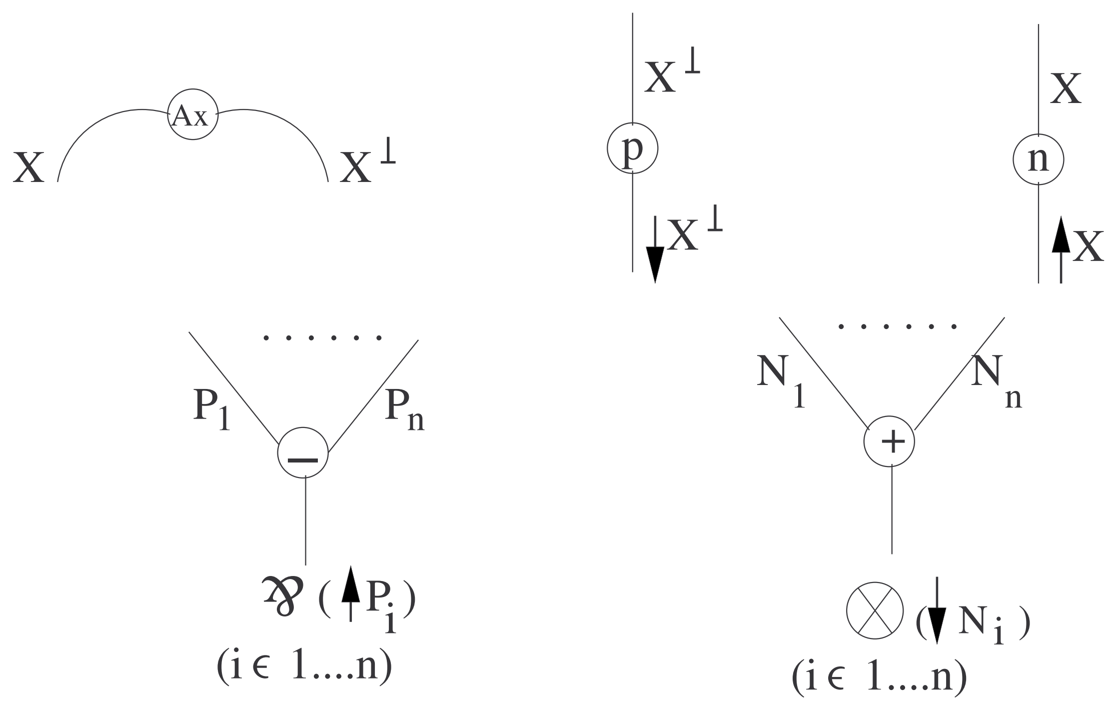
$$\frac{\vdash \Gamma,\ X}{\vdash \Gamma,\ \uparrow X}\ (n\text{-check}) \text{ if } X = \text{atom}^{+}$$

$$\frac{\vdash \Gamma,\ X^{\perp}}{\vdash \Gamma,\ \downarrow X^{\perp}}(p\text{-check}) \text{ if } X^{\perp} = \text{atom}^{-}$$

$$\frac{\vdash \Gamma_1,\ N_1 \ldots \vdash \Gamma_n,\ N_n}{\vdash \Gamma_1, \ldots, \Gamma_n,\ \otimes_{i \in I} \downarrow (N_i)}(+) \text{ if } N_{i \in \{1 \ldots n\}} \neq \text{atom}$$

$$\frac{\vdash \Gamma,\ P_1, \ldots P_n}{\vdash\ \Gamma,\ \parr_{i \in I}(\uparrow P_i)}(-) \text{ if } P_{i \in \{1 \ldots n\}} \neq \text{atom}$$

Simple proof structures



A simple proof structure has at most a negative terminal link

Proof net with jumps

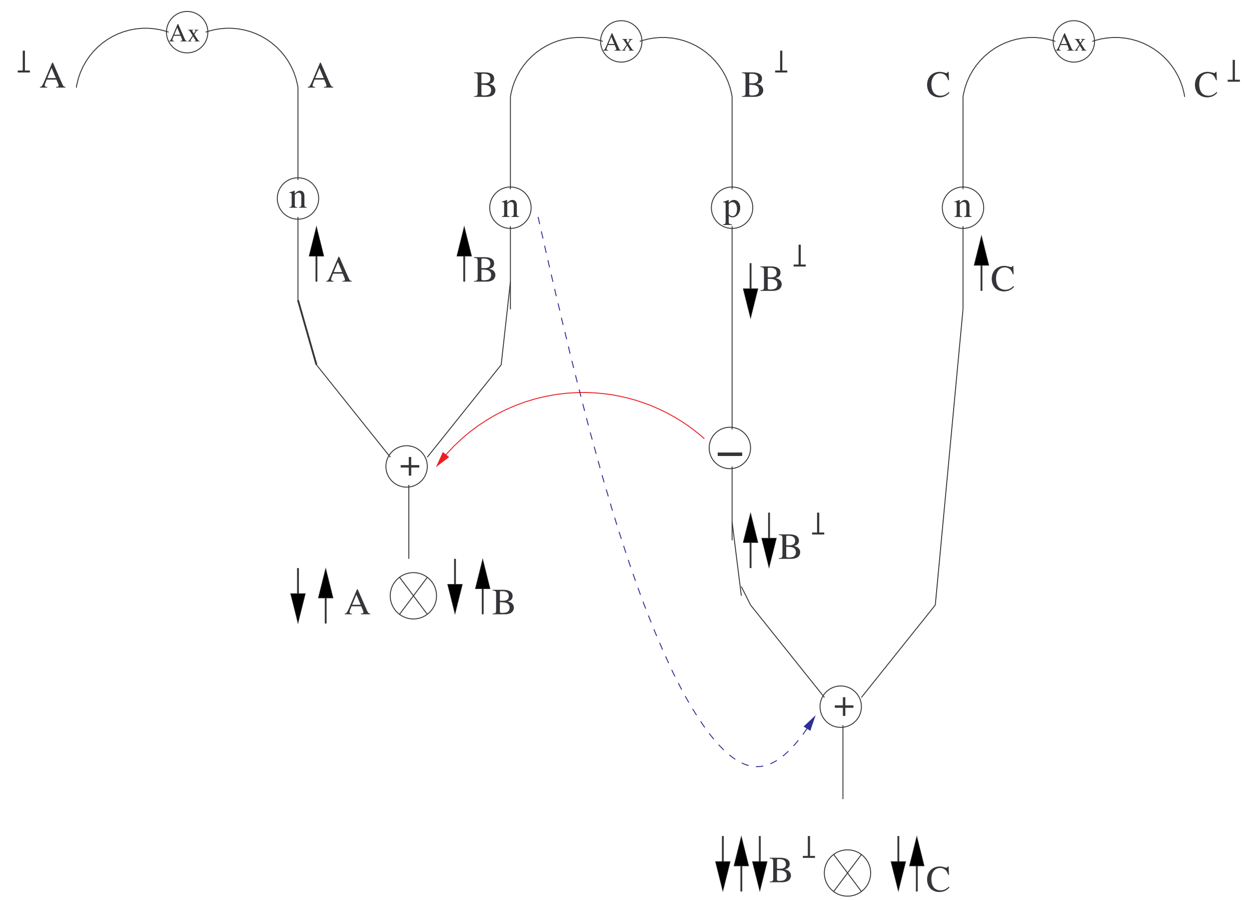
Definition 1 (Jumped proof structure) *A* jumped proof structure *(we abbreviate it with J.P.S.) is a S.P.S. to which have been added a certain number of not typed oriented edges called* jumps *from a negative to a positive link.*

We define switchings and correction graphs as usually: jumps are considered as premises of the negative links they emerge from.

Definition 2 (Jumped proof-net) *A* jumped proof net *(we abbreviate it with J.P.N.) is a J.P.S. R such that for every switching s of R, the correction graph $s(R)$ is acyclic and connected.*

Definition 3 (Simple proof net) *A* simple proof net *(we abbreviate it with S.P.N.) is a J.P.N. in which there aren't jumps.*

Example: how jumps increase sequentiality



Characterizing sequentiality :the order $\prec_R$

Given a J.P.N. R we associate to it an order $\prec_R$:

1. we reverse the orientation of all the edges of R which are not jumps;

2. we take the order defined by the transitive closure of the precedence relation induced by the graph.

A strict order r on a set A is arborescent when $\forall a, b$ if $\exists c$ s.t. $a < c \wedge b < c$ then $a < b \vee b < a$; an order r is a tree order when is arborescent and it has a minimum.

Arborisation lemma

Definition 4 (Saturation) *A J.P.N. R is* saturated *when for every negative link n and for every positive link p adding a jump between n and p creates a switching cycle or doesn't increase $\prec_R$.*

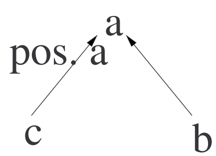
Lemma 1 (Arborisation) *Let R be a J.P.N. ; R is saturated iff $\prec_R$ is a tree order.*

The direction $\Leftarrow$ is trivial; we split the other direction in two parts:

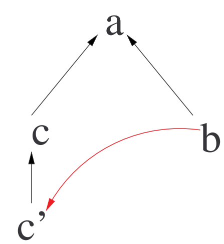
- *saturated $\Rightarrow$ arborescence;*
- *arborescence $\Rightarrow$ tree order.*

Proof part 1: saturated ⇒ arborescence

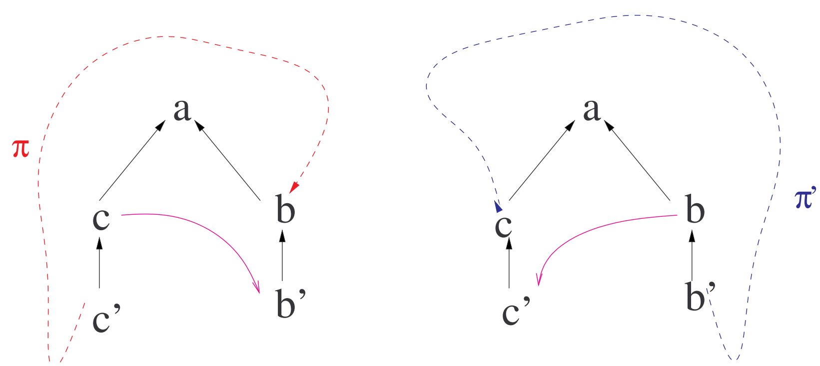
Not arborescent implies there exist a pos. a
b,c, neg such that:



1) b or c is a conclusion

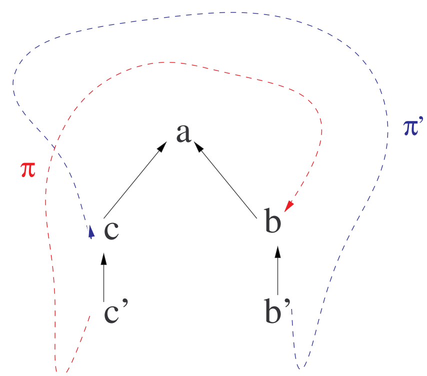


2) neither b or c are
conclusions

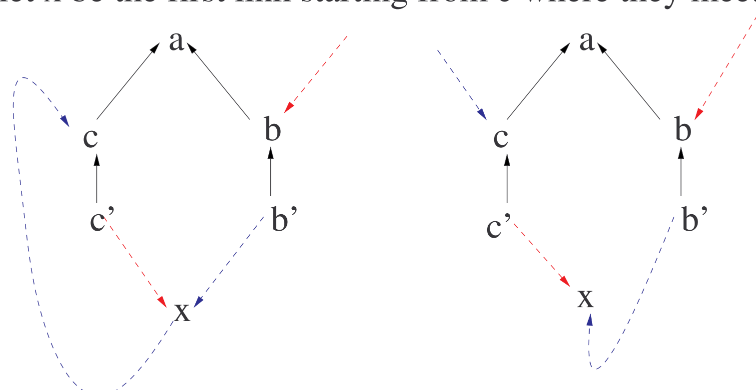


Proof part 2: saturated $\Rightarrow$ arborescence

1) π and π' are disjoint



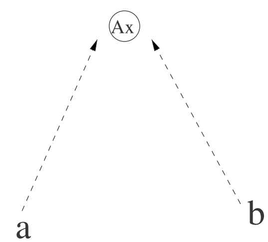
2) π and π' are not disjoint: let x be the first link starting from c where they meet



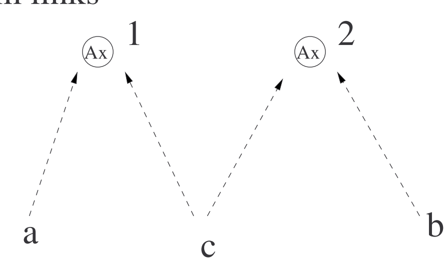
Proof part 3: arborescence ⇒ tree order

Let a,b be two minimals

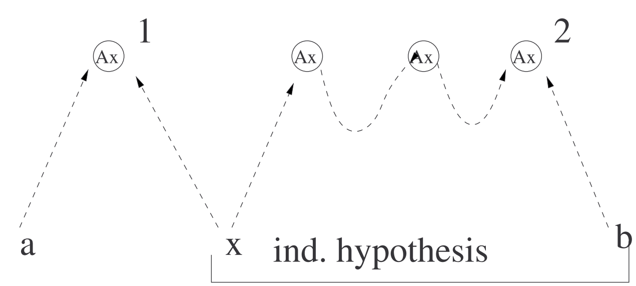
1)



2.1) by induction on the number n of axiom links on the path linking Ax 1 with Ax 2
n=0



2.2) n + 1



Eliminating the redundancies

Definition 5 *Given two J.P.N.* R, R', $R \equiv R'$ *iff their associated order* $\prec_R$ *and* $\prec_{R'}$ *are equal.*

The equivalence class of a J.P.N. R induced by $\equiv$ has a canonical representative which is called the *canonical jumped proof net of* R:

Definition 6 (Canonical proof net) *Given a J.P.N.* R *we call* canonical jumped proof net of R *(we denote it* $|R|$*) the J.P.N. obtained by erasing in* R *all the jumps which are transitive (a jump of* R *from* a *to* b *is transitive when erasing the jump* a *still precedes* b *in the order).*

Partial sequentialisation

$Jump(R)$**:** 1. add a correct jump, getting a J.P.N. R';

2. take the canonical J.P.N. $|R'|$ associated to R';

$DeJump(R)$**:** 1. take the canonical J.P.N. $|R|$ associated to R;

2. eliminate a jump, getting a J.P.N. R';

Theorem 1 (Partial sequentialisation) *Given any two J.P.N.* R, R' *s.t.* $R' = Jump^n(R)$ *for a certain* n, $DeJump^n(R') = R$.

A saturated proof net is a sequential proof

Lemma 2 (Splitting lemma) *Let R be a J.P.N. that has only terminal positive links and Ax link; if a $+$ link is the minimum of $\prec_R$,is splitting in R.*

Proposition 1 *Let R be a saturated J.P.N. :*

1. *if R has a negative conclusion w, w is minimum in $\prec_R$;*
2. *the graph $R_1, \ldots, R_n$ obtained from R erasing the minimum n of $\prec_R$ with its conclusion are saturated J.P.N.*

Proposition 2 *Let R be a saturated J.P.N. with a negative terminal link n; then n has a unique immediate successor in $\prec_R$.*

Full sequentialisation

Given a saturated J.P.N. R^J and a proof π , we associate to the order $\prec_{R^J}$ a proof $\pi^{\prec_{R^J}}$, and to π a S.P.N π^{des}.

Theorem 2 (Sequentialisation for simple proof nets) *Given a S.P.N. R, there exists a proof π of $MLL_{synt}^{\uparrow\downarrow}$ s.t. $\pi^{des} = R$.*

The proof is by induction on the number of links of R; we take any saturation R^J of R and we show that $(\pi^{\prec_{R^J}})^{des} = R$

Theorem 3 (Sequentialisation for jumped proof nets) *Given a J.P.N. R, there exists a proof π of $MLL_{synt}^{\uparrow\downarrow}$ s.t. $\exists n, Jump^n(\pi^{des}) = R$.*

It follows from the precedent theorem and the definition of $Jump$.

MLL

We define a traduction $()^*$ from the formulas of MLL to the formulas of $MLL{\uparrow\downarrow}_{synt}$ and a function f from links of MLL to links of $MLL{\uparrow\downarrow}_{synt}$.

Theorem 4 (Sequentialisation for MLL) *Given a proof net R of MLL there exists a proof π s.t. $\pi^{des} = R$.*

We follow the proof of Girard, except in the induction step, when R hasn't $\parr$ terminal links; in this case we found a splitting tensor by translating R in $f(R)$, getting a saturation R^J of $f(R)$; once we have a splitting tensor, we can proceed as in the proof of Girard, and we have done.

Future work

- Cut-elimination
- $MALL\ldots$